Parabolic mirrors

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Abstract

While it can be considered common knowledge that mirrors bundling incoming parallel rays into a unique focal point need to be parabolic in shape, it is less commonly known how to arrive at such a result. In this article we start with the desired condition on such a mirror and derive an equation for a function $f : \mathbb{R} \to \mathbb{R}$ describing the shape of it.

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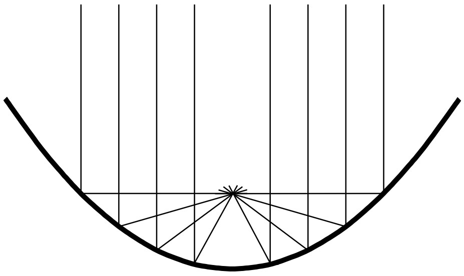

Figure 1: A parabolic mirror bundling incoming parallel rays into a focal point

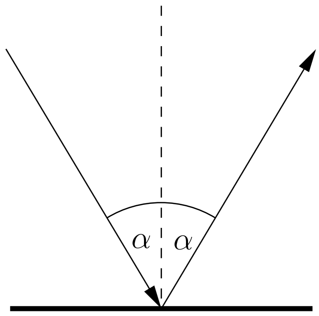

Figure 2: Reflection: The angle α of the incoming ray is equal to the angle of the reflected ray.

1 Introduction and notation

We consider parallel rays that are reflected on a mirror and ask what shape must the mirror have to bundle these rays into a unique focal point. Somewhat anticipating the result we refer to such mirrors as *parabolic mirrors* (see figure 1).

The rule how a ray is reflected is very simple: The angle of the incoming ray must be the same as the angle of the reflected ray (see figure 2). On a curved surface, this angle is measured against the line perpendicular to the tangent.

Without loss of generality we may assume that the parallel rays come from above and that the shape of the mirror is described by the graph of a function $f : \mathbb{R} \to \mathbb{R}$. Using analytic geometry we will arrive at a differential equation describing a class of functions having the desired property.

Our strategy for finding an equation for f will be the following: We will first consider full lines instead of rays and calculate their reflections on the graph of f. The resulting set of lines will be called the *reflection set* of f.

By analyzing this set we will arrive at conditions on f for having a unique focal point. These conditions will come in the form of an ordinary differential equation which we will solve by standard means (see also e.g. [Fur]).

Considering full lines instead of rays is a simplification which leads to solutions f which do not fulfill our initial requirements. We will do the necessary adjustments in the final section and arrive at the main theorem. –

Throughout this article we describe straight lines in $\mathbb{R}^2$ as the set of points (x, y) satisfying an equation $ax + by = c$ for a, b, $c \in \mathbb{R}$. Maybe the reader is more familiar with the functional description $y = mx+b$ where m is the slope and b is the y-intercept. This representation has the disadvantage, though, that it does not represent straight lines parallel to the y-axis and that it is difficult to apply transformations to it.

Note that the coefficients a, b of the equation $ax + by = c$ can be viewed as a vector $(a, b)^T$ being perpedicular to the line. In our calculations we won't impose any additional conditions on this vector; in particular we won't require it to be of length 1 or have a particular orientation. We refer to this representation as the *weak Hesse normal form* of a line L.

When calculating the reflection of an incoming ray we make use of the *dot product* of vectors in $\mathbb{R}^2$, this way avoiding having to deal with angles directly. We denote this product by $\langle\, , \,\rangle$. This notation is common in mathematics but less common in the applied sciences. This notation has the advantage, though, that it avoids amiguities between the dot product and ordinary multiplication in some cases.

2 The reflection set

We start with an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ and consider lines L_ε parallel to the y-axis, i.e. $L_\varepsilon = \{x = \varepsilon\}$ for some parameter $\varepsilon \in \mathbb{R}$. To each such line L_ε we will calculate its reflection L'_ε on the graph of f and call the set of all such reflected lines the *reflection set of f*.

We calculate L'_ε by transforming its normal vector, which we will denote by $\vec{n}'_\varepsilon$. For this, we use a formula derived in appedix A:

$$\vec{v}' = \frac{2\,\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|^2}\vec{u} - \vec{v}, \tag{1}$$

where $\vec{v}$ is the vector to be reflected and $\vec{u}$ is the vector defining the line on which to mirror $\vec{v}$.

In our application $\vec{v} = \vec{n}_\varepsilon$ is the normal of L_ε and $\vec{u} = \vec{n}_f$ if the normal defining the perpendicular at ε, i.e. a tangent vector on the graph of f (see figure 3).

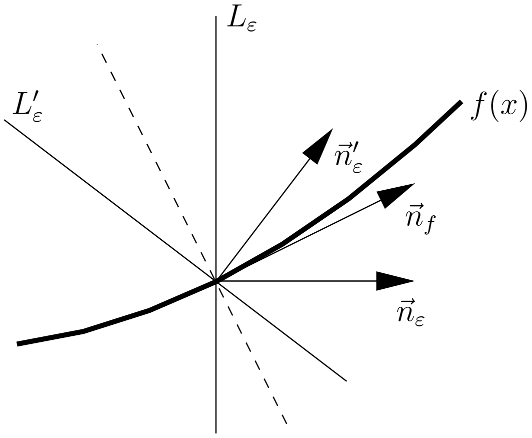

Figure 3: A line L_ε modelling an incoming ray is reflected on the graph of f

The normal vector of L_ε is given by

$$\vec{n}_\varepsilon = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The vector $\vec{n}_f$ defining the perpendicular at $(\varepsilon, f(\varepsilon))$ is given by

$$\vec{n}_f = \begin{pmatrix} 1 \\ f'(\varepsilon) \end{pmatrix}.$$

Using equation (1) we obtain the normal $\vec{n}'_\varepsilon$ defining the reflected line L'_ε:

$$\vec{n}'_\varepsilon = \frac{2\langle \vec{n}_\varepsilon, \vec{n}_f \rangle}{\|\vec{n}_f\|^2} \vec{n}_f - \vec{n}_\varepsilon.$$

Now $\|\vec{n}_f\|^2 = 1 + f'(\varepsilon)^2$ and $\langle \vec{n}_\varepsilon, \vec{n}_f \rangle = 1$. Therefore:

$$\vec{n}'_\varepsilon = \frac{2}{1 + f'(\varepsilon)^2} \begin{pmatrix} 1 \\ f'(\varepsilon) \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Using

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2}{1 + f'(\varepsilon)^2} \begin{pmatrix} (1 + f'(\varepsilon)^2)/2 \\ 0 \end{pmatrix}$$

we obtain

$$\vec{n}'_\varepsilon = \frac{2}{1 + f'(\varepsilon)^2} \begin{pmatrix} 1 - (1 + f'(\varepsilon)^2)/2 \\ 2f'(\varepsilon) \end{pmatrix} = \frac{1}{1 + f'(\varepsilon)^2} \begin{pmatrix} 1 - f'(\varepsilon)^2 \\ 2f'(\varepsilon) \end{pmatrix} \tag{2}$$

In order to obtain the (weak) Hesse normal form for L'_ε we need to find a constant c_ε, such that $L'_\varepsilon = \{\langle \vec{n}'_\varepsilon, \vec{x}\rangle = c_\varepsilon\}$. This constant is given by $\langle \vec{n}'_\varepsilon, p_\varepsilon\rangle$, where p_ε is the intersection of L'_ε with the graph of f, i.e. $p_\varepsilon := (\varepsilon, f(\varepsilon))$ We have

$$\begin{aligned}\langle \vec{n}'_\varepsilon, p_\varepsilon\rangle &= \frac{1}{1+f'(\varepsilon)^2}\left\langle \begin{pmatrix}1-f'(\varepsilon)^2\\ 2f'(\varepsilon)\end{pmatrix}, \begin{pmatrix}\varepsilon\\ f(\varepsilon)\end{pmatrix}\right\rangle \\ &= \frac{1}{1+f'(\varepsilon)^2}\left(\left(1-f'(\varepsilon)^2\right)\varepsilon + 2f'(\varepsilon)f(\varepsilon)\right).\end{aligned}$$

Therefore

Proposition. *The reflection set of a function $f : \mathbb{R} \to \mathbb{R}$ is given by $\mathcal{L} = \{L'_\varepsilon : \varepsilon \in \mathbb{R}\}$, where*

$$L'_\varepsilon = \left\{\left(1-f'(\varepsilon)^2\right)x + 2f'(\varepsilon)y = \left(1-f'(\varepsilon)^2\right)\varepsilon + 2f'(\varepsilon)f(\varepsilon)\right\} \qquad (3)$$

We conclude this section with an observation: If $f'(\varepsilon) = 0$, then $L'_\varepsilon = \{x = \varepsilon\}$, i.e. L'_ε is a line parallel to the y-axis. This means that if f' vanishes at multiple points, we obtain multiple reflected lines all being parallel to the y-axis. In such a scenario no unique focal point exists. We keep this observation for later reference:

Remark. If the function $f : \mathbb{R} \to \mathbb{R}$ has a unique focal point, then f' may vanish at most once.

3 Condition for a unique focal point

The condition on f for having a unique focal point is that all reflected rays L'_ε, $\varepsilon \in \mathbb{R}$, must intersect in a point $p_f \in \mathbb{R}^2$ which is not dependent on ε, i.e.:

$$\bigcap_{\varepsilon\in\mathbb{R}} L'_\varepsilon = \{p_f\}. \qquad (4)$$

When constructing the reflection set we made no assumptions on the function $f : \mathbb{R} \to \mathbb{R}$. However, the defining geometric property, namely having a unique focal point, is invariant under translation. That is, if f has a unique focal point, then the function $g(x) = f(x-a) + b$ will also have one for any a, $b \in \mathbb{R}$. Therefore, we can assume without loss of generality, that if a unique focal point exists, it lies in the origin $0 \in \mathbb{R}^2$, i.e. $p_f = 0$.

Remark. At this point it is import to keep the logic in mind that we are applying here. We *assume* that a unique focal point exists and based on that assumption we continue to find an equation for f. What if our assumption was wrong and there exists no such function f? We could still derive an equation for f but that equation would then be meaningless. Therefore, once we have derived an equation for f, we need to check that f indeed fulfills our requirement. –

For the reflection set this means that we have

$$L'_\varepsilon \cap \{x = 0\} = \left\{x = 0 \text{ and } 2f'(\varepsilon)y = \left(1 - f'(\varepsilon)^2\right)\varepsilon + 2f'(\varepsilon)f(\varepsilon)\right\}.$$

To obtain a condition on f, we need to solve for y in the above equation. For this, we have to assume that $f'(\varepsilon)$ is non-zero. With the remark of the previous section, we know that this can only happen at at most one point $\varepsilon_0 \in \mathbb{R}$. Exluding this point we have:

$$y = \frac{1}{2}\left(\frac{1}{f'(\varepsilon)} - f'(\varepsilon)\right)\varepsilon + f(\varepsilon) \overset{!}{=} 0, \quad \varepsilon \neq \varepsilon_0. \tag{5}$$

What does excluding ε_0 mean for our investigation? It means that once we have a potential solution for f, this solution is only valid on the domain $\mathbb{R}\backslash\{\varepsilon_0\}$, and that we must explicitly check that the reflected line at ε_0 does indeed intersect in the (hopefully) unique focal point.

Solving the right-hand side of equation (5) for $f(\varepsilon)$ and replacing ε by x in our notation, we obtain a differential equation for f:

Proposition. *The condition on $f : \mathbb{R} \setminus \{\varepsilon_0\} \to \mathbb{R}$ for having a unique focal point is:*

$$f = x \cdot \frac{1}{2}\left(f' - \frac{1}{f'}\right) \tag{6}$$

4 Solving the differential equation

Equation (6) is a d'Alembert differential equation (see appendix B for explanation and notation) with

$$F(p) = \frac{1}{2}\left(p - \frac{1}{p}\right) = \frac{p^2 - 1}{2p} \quad \text{and} \quad G(p) = 0.$$

Since F does not have a fixpoint, there is no trivial solution $f(x) = Cx$. For a non-trivial solution, we need to find $x = x(p)$ satisfying the following equation

$$x'(p) = \frac{F'(p)}{p - F(p)} x(p).$$

Calculating $F'(p) = (p^2+1)/2p^2$ and $p - F(p) = (p^2+1)/2$ the above equation simplifies to

$$x'(p) = \frac{1}{p}x(p).$$

This equation can be easily integrated to $x(p) = cp$ for some constant $c \neq 0$ (see for example [Fur]). We need to exclude $c = 0$ here because we need $x(p)$ to be invertible, so that we can derive $p = p(x)$ further down. Inserting this into equation (10) gives a solution for f with respect to p:

$$f(p) = cp \cdot \frac{1}{2}\left(p + \frac{1}{p}\right) = \frac{c}{2}p^2 + \frac{c}{2}.$$

Using $p = (1/c)x$ gives a solution with respect to x:

Proposition. *Equation* (6) *is solved by*

$$f(x) = \frac{1}{2c}x^2 + \frac{c}{2}. \tag{7}$$

for any $c \neq 0$.

At this point we need to return to the remark further above about the logic we are applying here. We derived the above equation for f based on the *assumption* that a function with a unique focal point exists. We now have to check that $f(x)$ as given in (7) indeed fulfills our requirement. For this, we simply need to insert f into equation (3) and check that the L'_ε intersect in $0 \in \mathbb{R}^2$. This is left as an exercise for the reader.

Furthermore, in the domain of f we left out the point ε_0 on which f' vanishes. Since $f'(x) = (1/c)x$, we see that $\varepsilon_0 = 0$. The corresponding reflected line $L'_0 = \{x = 0\}$ does indeed intersect the focal point $p_f = 0$. Therefore, we can extend the domain of f to $\mathbb{R}$.

5 Final result

When calculating the reflection set of f we used lines L_ε parallel to the y-axis. This way we considered rays coming from above and below simultaneously. In our initial setup, however, we only allow rays coming from above. The consequence of considering full lines L_ε in our calculations is that there are solutions f which do not satisfy our initial requirement (see figure 4). We will now remedy that.

To model an incoming ray coming from above we "cut" the lines L_ε and their reflections L'_ε at the graph of f. We do so by introducing the following set (see figure 5):

$$U_f^+ = \left\{(x, y) \in \mathbb{R}^2 : y \geq f(x)\right\}. \tag{8}$$

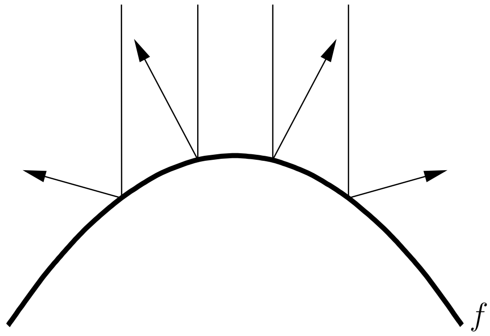

Figure 4: A solution f which does not fulfill the initial requirement. While f does have a unique focal point, it is "on the wrong side".

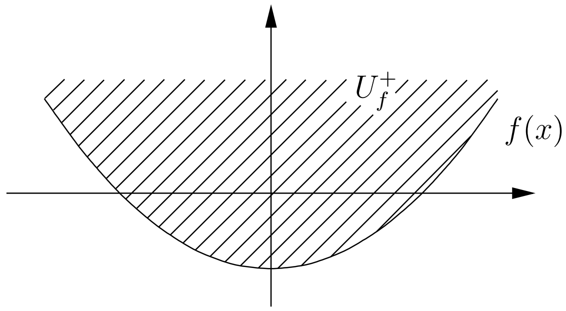

Figure 5: The area above the graph of f

The incoming ray is then

$$L_\varepsilon \cap U_f^+.$$

We now intersect the reflection set of f with U_f^+ and make use of the fact that the intersection of sets is associative:

$$\bigcap_{\varepsilon \in \mathbb{R}} \left(L'_\varepsilon \cap U_f^+\right) = \left(\bigcap_{\varepsilon \in \mathbb{R}} L'_\varepsilon\right) \cap U_f^+ = \{p_f\} \cap U_f^+.$$

Therefore, to check if f is a valid solution, we only need to check whether the focal point $p_f = (0, 0)$ is above the graph of f. Using equation (7), we see that this is the case for all $c > 0$.

To adapt f in equation (7) to more common notation, we set $a = 1/(2c)$ and arrive at the final theorem (see figure 6):

Theorem. *For any parameter $a > 0$ the function*

$$f(x) = ax^2 - \frac{1}{4a}$$

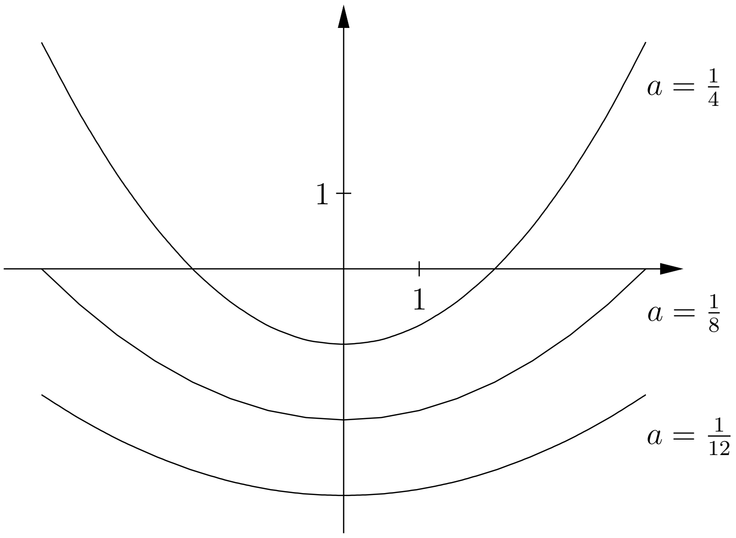

Figure 6: Parabolas $f(x) = ax^2 - 1/(4a)$ for various parameters a. All have the origin as their focal point. Notice how they "wander down" the y-axis as a decreases and they become flatter.

has a unique focal point at $(0, 0)$ *for rays coming from above.*

A Reflecting a vector on a straight line

We are going to derive a formula for reflecting a vector $\vec{v}$ on a line generated by a vector $\vec{u}$ (see figure 7). The key ingredient is a vector $\vec{x}$ that, when being added to $\vec{v}$, yields the *orthogonal projection* of $\vec{v}$ onto the line generated by $\vec{u}$. If we add this vector $\vec{x}$ twice to $\vec{v}$, we will obtain the reflection $\vec{v}'$.

Since $\vec{v} + \vec{x}$ shall by definition be the orthogonal projection of $\vec{v}$ onto $\mathbb{R}\vec{u}$, there has to be a scalar $\lambda \in \mathbb{R}$ such that $\vec{v} + \vec{x} = \lambda\vec{u}$, i.e. $\vec{x} = \lambda\vec{u} - \vec{v}$. We find λ by using that $\vec{x}$ shall be perpendicular to $\vec{u}$, i.e.

$$\langle \vec{u}, \vec{x} \rangle = \langle \vec{u}, \lambda\vec{u} - \vec{v} \rangle = \lambda \left\| \vec{u} \right\|^2 - \langle \vec{u}, \vec{v} \rangle \stackrel{!}{=} 0.$$

Therefore, λ is given by the equation

$$\lambda = \frac{\langle \vec{u}, \vec{v} \rangle}{\left\| \vec{u} \right\|^2}.$$

The reflected vector $\vec{v}'$ is thus:

$$\vec{v}' = \vec{v} + 2\vec{x} = \vec{v} + 2\left(\frac{\langle \vec{u}, \vec{v} \rangle}{\left\| \vec{u} \right\|^2}\vec{u} - \vec{v} \right) = \frac{2\langle \vec{u}, \vec{v} \rangle}{\left\| \vec{u} \right\|^2}\vec{u} - \vec{v}. \tag{9}$$

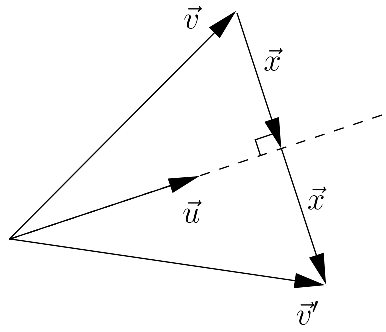

Figure 7: Reflecting a vector $\vec{v}$ on a line generated by $\vec{u}$

B D'Alembert's differential equation

The *D'Alembert differential equation* is a class of a non-linear, first-order ordinary differential equations of the following form:

$$f = xF(f') + G(f'), \tag{10}$$

where F and G are differentiable functions and f is the function we are looking for.

The first observation is that if F has a fixpoint C, i.e. $F(C) = C$, then $f(x) = Cx+G(C)$ gives a trivial solution of (10), since f' is simply a constant C.

Finding non-trivial solutions involves two tricks. The first is to introduce a new coordinate $p = f'$ which turns (10) into the following equation:

$$f(p) = x(p)F(p) + G(p). \tag{11}$$

This has not yet helped us much, because we do not know yet what p is. This is where a the second trick is applied: We differentiate (11) with respect to this new coordinate and obtain:

$$f'(p) = x'(p)F(p) + x(p)F'(p) + G'(p). \tag{12}$$

Now we observe that, by the chain-rule, we have:

$$f'(p) = \frac{df}{dp} = \frac{df}{dx}\frac{dx}{dp} = p\frac{dx}{dp} = px'(p).$$

Thus, replacing $f'(p)$ by $px'(p)$ in (12) and solving for $x'(p)$ we obtain

$$x'(p) = \frac{F'(p)}{p - F(p)}x(p) + \frac{G'(p)}{p - F(p)}. \tag{13}$$

This is a linear, inhomogeneous differential which is easier to solve than the original one. The solution is a function $x = x(p)$ which we need invert in order to obtain $p = p(x)$. The latter function can then be inserted into (11) to obtain the solution $f(x)$ of the original equation.

References

[Fur] Peter Furlan, *Das gelbe Rechenbuch 1–3*, Verlag Martina Furlan, 2012