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ONLY COUNTABLY MANY SIMPLY-CONNECTED LIE GROUPS ADMIT LATTICES

JÖRG WINKELMANN

A *lattice* in a locally compact topological group G is a discrete subgroup $\Gamma \subset G$ such that the quotient space G/Γ admits a G-invariant probability measure. This includes all discrete subgroups $\Gamma \subset G$ with compact quotient G/Γ. An important class of examples of lattices is given by arithmetic subgroups in linear-algebraic groups.

Lattices tend to strongly reflect the properties of the ambient group. For instance, if Γ_i $(i = 1, 2)$ are lattices in simply-connected semisimple or nilpotent real Lie groups G_i and $\Gamma_1 \simeq \Gamma_2$, then $G_1 \simeq G_2$. Moreover, any lattice Γ in a complex linear-algebraic group G is dense in the Zariski-topology.

In this spirit it is natural to assume that are only countably many non-isomorphic Lie groups admitting lattices, because lattices in Lie groups are finitely presentable and evidently there are only countably many non-isomorphic finitely presentable groups. We prove that indeed there are only countably many simply-connected real or complex Lie groups containing lattices. Since there do exist continuous families of non-isomorphic simply-connected Lie groups, this underlines that the existence of a lattice is a rather special property.

Theorem. *Let $k = \mathbb{R}$ or $k = \mathbb{C}$. Then up to isomorphism as k-Lie groups there exist only countably many simply-connected k-Lie groups admitting lattices (i.e. discrete subgroups such that the quotient admits an invariant probability measure).*

Remarks. – *The assumption of the Lie group being simply-connected is necessary, since every compact complex torus is an example of a complex Lie group containing the trivial subgroup $\{e\}$ as lattice.*

– *There do exist isomorphic lattices in non-isomorphic Lie groups. For instance, $\Gamma = (\mathbb{Z}^3, +)$ may be embedded as a lattice in both $G_1 = (\mathbb{R}^3, +)$ and*

$$G_2 = \left\{ \begin{pmatrix} 1 & t & & & \\ & 1 & & & \\ & & \cos t & \sin t & x \\ & & \text{-}\sin t & \cos t & y \\ & & & & 1 \end{pmatrix} : t, x, y \in \mathbb{R} \right\}$$

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1. Reduction to the real case

For a complex Lie group the existence of a lattice has nothing to do with the complex structure on G. If a complex Lie group G contains a lattice, so does every complex Lie group which is isomorphic to G as a topological group. Hence it is natural to start our considerations by considering the following question: *Given* *a complex Lie group G, how many non-isomorphic complex Lie groups can there exist which are isomorphic to G as topological groups ?*

Proposition 1. *Let G be a simply-connected real Lie group. Then there exist only finitely many non-isomorphic complex Lie groups $(H_i)_{i\in I}$ which are isomorphic to G as topological groups (resp. real Lie groups).*

Proof. We begin the proof by the following auxiliary result on direct summands of Lie algebras. (A Lie subalgebra $\mathbf{h}$ of a Lie algebra $\mathbf{g}$ is called a *direct summand* if there exists another Lie subalgebra $\mathbf{i}$ such that $\mathbf{g} = \mathbf{h} \oplus \mathbf{i}$.)

Lemma 1. *Let k be a field and $\mathbf{g}$ a finite-dimensional k-Lie algebra.*

Then there exist only finitely many direct summands up to isomorphism as Lie algebra.

Proof. Let $d \in \mathbb{N}$. The set of all d-dimensional Lie subalgebras of $\mathbf{g}$ carries the structure of a closed subvariety V of a Grassmann variety in a natural way. For every direct summand $\mathbf{h}$ there is a projection $\pi : \mathbf{g} \to \mathbf{h}$ with a kernel $\ker \pi = \mathbf{c}$. If $\mathbf{i}$ is a Lie subalgebra of $\mathbf{g}$ with $\dim \mathbf{h} = \dim \mathbf{i}$ and $\mathbf{i} \cap \mathbf{c} = \{0\}$, then π induces a Lie algebra isomorphism between $\mathbf{h}$ and $\mathbf{i}$. Thus for every direct summand $\mathbf{h}$ there is an open neighbourhood U of $[\mathbf{h}]$ in V such that $\mathbf{i} \simeq \mathbf{h}$ for every $[\mathbf{i}] \in U$. Now the lemma follows from quasi-compactness of Zariski topology. □

We now show that this lemma implies the proposition. By the assumption of simply-connectedness it suffices to prove the corresponding statement for Lie algebras. Let $\mathbf{g}_i$ be a family of complex Lie algebras such that the underlying real Lie algebras are all isomorphic to a given real Lie algebra $\mathbf{h}$. In this case $\mathbf{h} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbf{g}_i \oplus \overline{\mathbf{g}_i}$ for every i. In particular $\mathbf{g}_i$ is a direct summand of $\mathbf{h} \otimes_{\mathbb{R}} \mathbb{C}$. Hence we may invoke the lemma in order to conclude the proof of the proposition. □

2. Reduction to the solvable case

The reduction of the general case to the solvable case consists of two steps. First we show that if a Lie group admits a lattice, so does its radical. Second we show that there are only countably many non-isomorphic Lie groups with prescribed radical.

Proposition 2. *Let G be a connected real Lie group with radical R. Assume that G contains a lattice.*

Then R also contains a lattice.

Proof. Let Γ be a lattice in G. Let C denote the maximal normal connected compact semisimple Lie subgroup of G. The projection map $\pi : G \to G/C$ is proper, hence $\pi(\Gamma)$ is a lattice in G/C. The intersection $R \cap C$ is finite, hence π restricts to a finite map from R onto $\pi(R)$. By construction G/C fulfills the condition of Cor. 8.28 of [3] [1]. Hence $\pi(\Gamma) \cap \pi(R)$ is a lattice in $\pi(R)$. It follows that $R \cap \pi^{-1}(\pi(\Gamma))$ is a lattice in R. □

Proposition 3. *Let R be a solvable real Lie group. Then there exist only countably many non-isomorphic simply-connected real Lie groups with R as radical.*

[1] The statement of Cor. 8.28 in [3] is correct, although the proof is not. See [4] for details.

Proof. If G is a simply-connected Lie group with radical isomorphic to R, then G is isomorphic to a semidirect product of a semisimple real Lie group S and the radical R which is given by a Lie group homomorphism $\rho : S \to AutR$ (Levi-Malcev-decomposition). Let U be a maximal connected semisimple Lie subgroup of $AutR$. We may assume $\rho(S) \subset U$, since the maximal connected semisimple Lie subgroups of a given Lie group are conjugate to one another. Now the proposition follows from the fact that there are only countably many non-isomorphic semisimple Lie groups and that there are only countably many non-conjugate Lie group homomorphisms between two given semisimple real Lie groups. □

3. The real solvable case

For a simply-connected solvable real Lie group our theorem follows from the proposition below which in turn is based on a arithmeticity result going back to Mostow.

Proposition 4. *Let G be a simply-connected solvable real Lie group and Γ a lattice in G.*

Then there exists a natural number n and finitely generated subgroups $\Lambda_1, \Lambda_2 \subset GL(n, \mathbb{Z})$ with the following properties:

1. *Λ_1 is a normal subgroup of Λ_2,*
2. *Let U denote the group of $\mathbb{R}$-rational points of the Zariski-closure of Λ_1 and H the connected component of the normalizer of $U \cdot \Lambda_2$ in $GL(n, \mathbb{R})$.*

   Then $U \cdot \Lambda_2$ is closed and the quotient Lie group $H/(U \cdot \Lambda_2)$ contains a compact Lie subgroup K such that $G \simeq \tau^{-1}(K)$ where $\tau : H \to H/(U \cdot \Lambda_2)$ is the natural projection.

Corollary. *Up to Lie group isomorphism there exist only countably many simply-connected solvable real Lie groups admitting a lattice.*

Proof. Evidently there are only countably many choices for n, Λ_1 and Λ_2. Thus it suffices to show that there are only countably many choices for K.

Claim. *Let G be a real Lie group. Up to conjugacy there are only countably many compact connected solvable Lie subgroups.*

This holds, because every compact connected solvable Lie group is a torus, all maximal tori are conjugate and a torus contains only countably many subtori. □

Proof. We start with the following result of Mostow-Raghunathan.

Claim. *Let G be a simply-connected solvable real Lie group, N its nilradical and Γ a lattice in G.*

Then there exists an injective Lie group homomorphism $i : G \to GL(n, \mathbb{R})$ (for some n) such that $i(\Gamma) \subset GL(n, \mathbb{Z})$. Moreover every element in $i(N)$ is unipotent.

This is Th.4.34 on p.71 of [3] except for the claim on the unipotency of $i(N)$. But this claim follows easily from the details of the proof as given in pp.72–76 in [3].

We define $\Lambda_2 = i(\Gamma)$ and $\Lambda_1 = i(\Gamma \cap N)$. Now $\Gamma \cap N$ is a lattice in N. Hence both Λ_i are finitely generated. Since $i(N)$ is unipotent, it furthermore follows that $U = i(N)$ is algebraic and equals the Zariski-closure of $i(N \cap \Gamma) = \Lambda_1$. The group $U \cdot \Lambda_2$ is closed, because the orbits of the natural left N-action on G/Γ are compact and therefore are in particular closed. Thus the normalizer H_0 of $U \cdot \Lambda_2$ in $GL(n, \mathbb{R})$

is a closed real Lie subgroup. Evidently $U = i(N)$ contains the commutator group of $i(G)$. Hence $i(G) \subset H_0$. Moreover $i(G)/U \cdot \Lambda_2$ is compact, because $G/\Gamma \simeq i(G)/\Lambda_2$ is compact. Hence the proposition. $\square$

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