Geometric Complex Analysis
edited by Junjiro Noguchi *et al.*
World Scientific, Singapore, 1995
pp.1– 99

Complex parallelizable manifolds

JOERG WINKELMANN

Mathematisches Institut
Ruhr-Universität Bochum
44780 Bochum
Germany

1. Introduction

In this note we give a survey of our recent results on complex-parallelizable manifolds, most of which are contained in [18].

For our purposes it is useful to call a complex manifold *parallelizable* if it is biholomorphic to a quotient of a complex Lie group G by a discrete subgroup Γ.

By a result of Wang ([16]) for a compact complex manifold this condition is equivalent to the assumption that the tangent bundle is holomorphically trivial.

A compact complex parallelizable manifold is Kähler if and only if G is commutative, i.e. if and only if it is a torus.

2. The structure of nilmanifolds

Here we describe arithmeticity results for complex nilmanifolds obtained in [18,19]. A *nilmanifold* X is a compact complex-parallelizable manifold with nilpotent fundamental group, i.e. a manifold isomorphic to a quotient of nilpotent complex Lie group G by a discrete subgroup Γ. Such a nilmanifold is a tower of torus principal bundles [7]. Our principal result concerning arithmeticity of nilmanifolds states that the tori occuring in this tower are arithmetic in the following sense: They are direct products of simple tori with complex multiplication. Here a simple torus T is said to have *complex multiplication* if the endomorphism algebra $End_{\mathbb{Q}}(T)$ contains a commutative subalgebra K with $dim_{\mathbb{Q}}(K) = 2\,dim_{\mathbb{C}}(T)$. This is known to be equivalent to the assumption that there exists a totally complex numberfield K of degree $2g$ over $\mathbb{Q}$ and a choice of g embeddings $\sigma_i : K \to \mathbb{C}$ (pairwise disjoint and non-conjugate) such that T is isogenous to $\mathbb{C}^g/\sigma(\mathcal{O}_K)$ (where $\mathcal{O}_K$) denotes the ring of algebraic integers of K).

In the converse direction, we can show that every torus with complex multiplication occurs in the tower of a nilmanifold in a non-trivial way.

Tori with complex multiplication are rather special tori. For instance, a one-dimensional torus $\mathbb{C}/\langle 1, \tau\rangle$ has complex multiplication if and only if $\tau^2 \in \mathbb{Q} + \tau\mathbb{Q}$. However, a torus with complex multiplication is not necessarily an abelian variety. In fact, for every totally complex number field K which is not a *CM*-field there exists a torus T with complex multiplication by K such that T admits no non-constant meromorphic functions at all. (A totally complex number field K is called a *CM*-field, if it contains a totally real subfield K^+ with $[K : K^+] = 2$.) Using this fact one can deduce the existence of non-trivial nilmanifolds wwithout non-constant meromorphic functions. Now we present the main result.

Theorem 1. *Let G be a connected, simply-connected complex nilpotent Lie group and Γ a discrete subgroup such that $X = G/\Gamma$ is compact. Let G^k denote the descending central series (i.e. $G^1 = G$, $G^{k+1} = [G, G^k]$) and C_k the ascending central series (i.e. $C_0 = \{e\}$, $C_{k+1} = \{c : cgc^{-1}g^{-1} \in C_k\ \forall g \in G\}$.)*

Let
$$X = X^n \xrightarrow{T_n} X^{n-1} \xrightarrow{T_{n-1}} \dots \xrightarrow{T_2} X^1 = T_1 = Alb(X)$$
denote the tower of torus principal bundles associated to the descending central series, (i.e. $X^n = G/G^{n+1}\Gamma$, $T_n = G^n/G^{n+1}\Gamma \cap G^n$) and
$$X = X_0 \xrightarrow{S_1} X_1 \xrightarrow{S_2} \dots \xrightarrow{S_{n-1}} X_{n-1} = S_n$$
the tower of torus principal bundles associated to the ascending central series. (It is well-known [7] that all the $G^k\Gamma$ and $C_k\Gamma$ are closed in G.)

Then for $k > 1$ both S_k and T_k are isogenous to a product of simple tori with complex multiplication.

If we impose a non-degeneracy condition on the nilmanifold, we can deduce some conditions for $Alb(X)$.

For the statement of this result we need to explain the notion of a *decomposition series* for a torus T. This is a series of subtori $\{e\} = T_0 \subset T_1 \subset \ldots \subset T_m = T$ such that all the quotients T_{k+1}/T_k are positive-dimensional and simple. These quotients T_{k+1}/T_k are called *simple factors* for this composition series. Up to ordering and isogenies, they are independent of the particular choice of a composition series, i.e. depend only on T. If T is algebraic, then it is isogenous to a direct product of these simple factors.

Theorem 2. *Assume the notations of the above theorem.*

Assume that $Z = C_1 \subset G' = G^1$. Then the composition series both for $Alb(X)$ and for any torus embedded in X as complex submanifold has the property that every simple factor admits complex multiplication.

If in addition $C \subset G'$ with $C = \{c \in G : cgc^{-1}g^{-1} \in G^3\}$, then $Alb(X)$ is isogenous to a product of simple tori with complex multiplication.

The condition $C \subset G'$ holds in particular in the following two cases:
(i) G is a generalized Heisenberg group, i.e. $Z = G'$.
(ii) G is a maximal unipotent subgroup in a complex semisimple Lie group.

(In the second case $C \subset G'$ can be verified easily using the theory of root systems for semisimple Lie groups.)

Example. *Let G be a simply-connected nilpotent complex Lie group G with one-dimensional center Z. (This implies $G' = Z$.) Let Γ be a discrete cocompact subgroup. Then $X = G/\Gamma \xrightarrow{C} G/G'\Gamma = Alb(X)$ is the Albanese, and our theorem implies that $C = G'/(G' \cap \Gamma)$ is an elliptic curve with complex multiplication and $Alb(X)$ is isogenous to a direct product of copies of C.*[1]

We have also a result in the converse direction, confirming that every torus with complex multiplication occurs in such a way.

Theorem 3. *Let $\mathbf{g}$ be a nilpotent $\mathbb{Q}$-Lie algebra, K a totally complex number field, $2g = [K : \mathbb{Q}]$.*

---

[1] The article *On the Picard Group of a compact complex nilmanifold* in the *Rocky Mt. J. Math.* 17 (1987), 65–76 contains some alleged examples of nilmanifolds which would provide counterexamples to this conclusion. These examples are based on the assumption that certain subgroups (given in terms of their generators) are discrete. But actually they fail to be discrete.

Then there exists a simply-connected nilpotent complex Lie group G, and a discrete cocompact subgroup Γ such that $\mathcal{L}ie(G)$ is isomorphic to a direct product of g copies of $\mathbf{g} \otimes_{\mathbb{Q}} \mathbb{C}$ and every torus $G^k/(G^{k+1}\Gamma)$ is a direct product of tori with complex multiplication by K.

If K is not a CM-field, then G and Γ may be choosen in such a way that none of the tori $G^k/(G^{k+1}\Gamma)$ admits a non-constant meromorphic function.

Considering number fields which are neither totally real nor totally complex, one can generalize this method of construction in order to obtain interesting non-compact nilmanifolds X such that every holomorphic function on X is constant, but every compact analytic subset is finite.

We have another type of converse to Theorem 1 which characterizes nilmanifolds among solvmanifolds.

Theorem 4. *Let G be a simply-connected solvable complex Lie group, Γ a discrete cocompact subgroup. Assume that the center Z is contained in the commutator group G'.*

Then G is nilpotent if and only if every simple factor in a decomposition series of $Alb(X) \simeq G/G'\Gamma$ has complex multiplication.

The reason for this phenomenon is based on the fact that a non-nilpotent solvable Lie group always involves the multiplicative group and that discrete cocompact subgroups in such groups enjoy a form of "multiplicative" arithmeticity which is incompatible with the kind of arithmeticity occuring in the nilpotent case. This can be made precise in the following way.

Theorem 5. *Let G be a simply-connected solvable complex Lie group, N its nilradical, Γ a discrete cocompact subgroup in G, $T = G/N\Gamma$. Let $T = \mathbb{C}^g/\Lambda$.*

Then there exists a homomorphism of complex Lie groups $\phi : \mathbb{C}^g \to (\mathbb{C}^*)^k$ with discrete kernel such that $\phi(\Lambda)$ is contained in the $\overline{\mathbb{Q}}$-rational points, i.e. $\phi(\Lambda) \subset \left(\overline{\mathbb{Q}}^*\right)^k$.

In order to verify that this property is incompatible with the arithmeticity properties of a nilmanifold, we will deduce the following result from Baker's theorem.

Proposition 1. *Let Λ be a lattice in $\mathbb{C}^g$ such that $\Lambda \subset \overline{\mathbb{Q}}^g$. Let $\phi : \mathbb{C}^g \to (\mathbb{C}^*)^k$ be a complex Lie group homomorphism such that $\phi(\Lambda) \subset (\overline{\mathbb{Q}}^*)^k$.*

Then ϕ is constant.

For investigating the structure of tori occuring in the tower of a nilmanifold, it is essential to use the structure for tori. For abelian varieties, there is a very satisfying structure theory. However, we need a structure theory for arbitrary tori, not only the algebraic ones. Many results for abelian varieties do not hold for arbitrary tori. Most important, an arbitrary torus is not necessarily isogenous to a product of simple tori. Thus we first have to develope some structure theory for non-algebraic tori.

One tool is the already mentioned notion of a composition series.

Proposition 2. *Every torus T admits a composition series, i.e. an ascending sequence of subtori $\{e\} = T_0 \subset \ldots T_m = T$ such that all the quotient $S_k = T_{k+1}/T_k$ are positive-dimensional and simple. The sequence S_k is called the sequence of simple factors for this composition series.*

If there exists another composition series, the sequence of simple factors differs only by ordering and isogenies.

If there is a surjective morphism of tori $T \to T'$ or an injective morphism of tori $T' \to T$, then the sequence of simple factors for T' is contained in the sequence of simple factors for T (up to ordering and isogenies).

An algebraic torus always equals the sum of all its simple subtori. In contrast, an arbitrary torus is generated by all its irreducible subtori. Here a torus T is called irreducible, if it is not generated by its proper subtori. We prove that irreducible non-simple tori can not occur in our context. The reason is the following:

Proposition 3. *Let $T = \mathbb{C}^g/\Lambda$ be an irreducible torus, E a commutative subalgebra of $End_{\mathbb{Q}}(T)$. Assume that $\Lambda \otimes \mathbb{Q}$ is a principal E-module, i.e. generated as E-module by one element.*

Then T is simple and E is a number field.

For a nilpotent group, the commutator map $\zeta(g,h) = ghg^{-1}h^{-1}$ is especially useful. In particular it yields a $\mathbb{Z}$-bilinear map $G/G' \times G/G' \to G'/G^2$. This enables us to conclude that the tori we encounter will have large endomorphism algebras. Furthermore these algebras must contain large commutative subalgebras. This follows from an auxiliary, purely algebraic result on bilinear forms on modules. Then the above stated result implies that all tori occuring in the tower of a nilmanifold must be semisimple, i.e. isogenous to a direct product if simple tori. Moreover these simple tori must have complex multiplication.

3. Deformations of nilmanifolds

As outlined in the previous section, the tori occuring in the torus bundle tower of a non-degenerate nilmanifold are necessarily arithmetic in the sense of complex multiplication. Naturally, this rise the question whether nilmanifolds are globally complex-arithmetic in the following sense.

Definition. *A complex-parallelizable manifold X is called complex-arithmetic if there exist an index set I, totally imaginary number fields K_i, algebraic K_i-groups G_i, types $\sigma^{(i)} = (\sigma_1^{(i)}, \ldots, \sigma_g^{(i)}) : K_i \hookrightarrow \mathbb{C}^{g_i}$ such that $X \simeq \Pi_{i \in I}{}^{\sigma^{(i)}} G_i(\mathbb{C})/\Gamma$ where Γ is a subgroup commensurable to $\Pi_{i \in I} \sigma^{(i)}(\mathcal{O}_{K_i})$.*

If the question is put in this simple form, the answer is clearly NO, because evidently there are no continuous families of complex-arithmetic manifolds whereas complex nilmanifolds always can be deformed as follows:

Proposition 4. *Let G a complex simply-connected nilpotent Lie group, Γ a discrete cocompact subgroup, $Z_{\mathbb{R}}$ a totally real connected Lie subgroup of the center of G with $Z_{\mathbb{R}} \subset [G, G]$. and $\rho : \Gamma \to Z_{\mathbb{R}}$ a group homomorphism.*

Then

$$\Gamma^\rho = \{\gamma \cdot \rho(\gamma) : \gamma \in \Gamma\}$$

is again a discrete cocompact subgroup, and G/Γ is biholomorphic to G/Γ^ρ if and only if $\rho \equiv 0$.

Since $Hom(\Gamma, Z_{\mathbb{R}})$ is connected, this guarantees the existence of deformations of nilmanifolds.

Thus one can only hope for arithmeticity up to deformation. We are able to prove this under an additional assumption.

Definition. *Let $\mathbf{g}$ be a nilpotent Lie algebra, $\mathbf{g}^k$ the (descending) central series. Then*

$$\hat{\mathbf{g}} = \oplus_{k \geq 1} \mathbf{g}^k / \mathbf{g}^{k+1}$$

is called the associated graded Lie algebra.

Theorem 6. *Let $X = G/\Gamma$ be a complex nilmanifold and assume that $\mathcal{L}ie(G)$ is isomorphic to its associated graded Lie algebra.*

Then X is a deformation of a complex-arithmetic manifold.

Unfortunately there exist nilpotent Lie algebras which are not isomorphic to their associated graded Lie algebra (see [3]).

4. Vector Bundles

In this section we are concerned with vector bundles over complex parallelizable manifolds. Naturally the most easily accessible vector bundles are those which are *homogeneous*, i.e. vector bundles $E \to X = G/\Gamma$ such that $g^*E \simeq E$ for all $g \in G = \mathrm{Aut}_{\mathcal{O}}(X)^0$. This is equivalent to the assumption that there exists some extension group of G whose action on X can be lifted to the action of E. Furthermore, over a compact complex parallelizable manifold, a vector bundle is homogeneous if and only if it admits a holomorphic connection.

A vector bundle is *flat* if there exists a holomorphic connection with vanishing curvature. This is equivalent to the assumption that it can be defined by a representation of the fundamental group. For a compact complex parallelizable manifold X it is furthermore equivalent to the assumption that the action of the universal covering group of $\mathrm{Aut}_{\mathcal{O}}(X)^0$ on X lifts to an action on the vector bundle.

Certain homogeneous vector bundles are automatically flat.

Theorem 7. *Let X be the quotient manifold of a simply-connected complex Lie group G and a discrete cocompact subgroup Γ. Let E be a homogeneous holomorphic vector bundle over X*

Then E is flat if one of the following conditions are fulfilled:

(i) G is semisimple;
(ii) E is a vector bundle of rank two and the radical R_G of G is nilpotent;
(iii) E is a line bundle.

For the special case where X is a torus this is a classical result of Matsushima [8].

For $X = G/\Gamma$ with G semisimple without $SL_2(\mathbb{C})$-factors we have a classification of homogeneous vector bundles.

Theorem 8. *Let G be a simply-connected semisimple complex Lie group, Γ a discrete cocompact subgroup, $X = G/\Gamma$. Assume that no direct factor of G is isomorphic to $SL_2(\mathbb{C})$.*

Then there is a one-to-one correspondence between indecomposable homogeneous vector bundles over X and pairs (ϕ, ζ) where ϕ is an irreducible holomorphic representation of G and $\zeta : \Gamma \to GL(V)$ is an irreducible representation of Γ such that $\zeta(\Gamma)$ is a relatively compact subgroup of $GL(V)$.

The correspondence is given in the following way: The vector bundle associated to (ϕ, ζ) is the flat bundle given by the representation $\rho = (\bar{\phi}|_\Gamma) \otimes \zeta$ of *the fundamental group* $\pi_1(X) \simeq \Gamma$. *(*$\bar{\phi}$ *is the antiholomorphic representation obtained from* ϕ *by complex conjugation.)*

An arbitrary homogeneous vector bundle E *is in a a unique way a direct sum* $E = \oplus_i E_i$ *of indecomposable homogeneous vector bundles* E_i.

Of course, the set of irreducible representations of Γ with relatively compact image depends on Γ. It includes in particular the irreducible representations with finite images. There are always plenty such representations, because Γ is residually finite as a finitely generated linear group. If Γ is *complex-arithmetic* as defined in I.7, then every representation of Γ with relatively compact image has in fact finite image. In general, superrigidity results (see [9]) allow a good description of representations with relatively compact image.

One can prove that for fixed Γ and fixed $n \in \mathbb{N}$ there exist only finitely many non-conjugate representations $\zeta : \Gamma \to GL_n(\mathbb{C})$ with relatively compact image. It follows that for fixed $X = G/\Gamma$ and fixed $n \in \mathbb{N}$ there exists only finitely many homogeneous vector bundles of rank $\leq n$ over X up to isomorphism.

For arbitrary G we have a criterion to determine which representations of the fundamental group yield holomorphically non-trivial bundle.

Here some word of caution is in order. Given a compact manifold, any non-trivial representation of the fundamental group yields a bundle with flat connection. There exists a trivialization *preserving the given flat connection* if and only if the representation is trivial. Nevertheless for non-trivial representations it is possible that there exists a trivialization of the induced bundle as holomorphic bundle. For example, consider line bundles over non-compact Riemann surfaces X. For every non-compact Riemann surface $Pic(X) = H^1(X, \mathcal{O}^*) = 0$, although $Hom(\pi_1(X), \mathbb{C}^*) \neq 1$ for every Riemann surface which is not simply-connected.

For parallelizable manifolds there is an exact criterion determining whether a flat bundle is holomorphically trivial. *Let* E *be a vector bundle* E *over a compact complex parallelizable manifold* $X = G/\Gamma$ *given by a representation of the fundamental group* $\rho : \pi_1(X) \simeq \Gamma \to GL(V)$. *Then* E *is trivial as holomorphic vector bundle if and only if* ρ *can be extended to a holomorphic representation of* G.

This implies the expected statement for line bundles. *Let* $X = G/\Gamma$ *be a compact complex parallelizable manifold. Assume that no direct factor of* G/R_G *is isomorphic to* $SL_2(\mathbb{C})$. *Then there is a one-to-one correspondence between homogeneous line bundles and* $Hom(\Gamma, S^1)$ *(with* $S^1 = \{z \in \mathbb{C}^* : |z| = 1$*).*

For general compact complex manifolds, vector bundles arising from representations of the fundamental group having bounded image are of a particularly simple nature. *Let X be a compact complex manifold, $\rho_i : \pi_1(X) \to GL(V)$ $(i = 1, 2)$ representations such that the images $\rho_i(\pi_1(X))$ are relatively compact subsets in $GL(V)$. Then the induced vector bundles are isomorphic as holomorphic vector bundles if and only if the ρ_i are conjugate (by an element in $GL(V)$).*

In particular it follows that, if $\rho : \pi_1(X) \to GL(V)$ a representation of the fundamental group with image $\rho(\pi_1(X)) \neq \{e\}$ finite, then the induced vector bundle is not holomorphically trivial.

We may use our classification of homogeneous vector bundles over parallelizable manifolds in order to answer a question of Bănică. He asked whether every non-trivial compact complex manifold admits a a non-trivial holomorphic vector bundle. Evidently a non-parallelizable compact complex manifold admits a non-trivial vector bundle, namely the tangent bundle. Hence it suffices to consider parallelizable manifolds and we may deduce the following result.

Theorem 9. *Let X be a compact complex manifold of dimension $n > 0$.*

Then there exists a non-trivial holomorphic vector bundle on X of rank not exceeding n.

(This is an improvement of the result published in [17]).

Flat vector bundles are close to being topologically trivial. The presence of a flat connection for a vector bundle E implies that all the Chern classes $c_k(E) \in H^{2k}(X, \mathbb{R})$ vanish. Furthermore, for every flat line bundle L over a compact complex manifold X, there exists a finite unramified covering $\pi : X_1 \to X$ such that $\pi^* L$ is topologically trivial. We prove similar results for flat vector bundles over certain parallelizable manifolds. These results even extend to homogeneous bundles, because (as mentioned above) in many cases homogeneous bundles over parallelizable manifolds are automatically flat.

Theorem 10. *Let E be a homogeneous vector bundle over a compact parallelizable manifold $X = G/\Gamma$. Then the Chern classes $c_k(E) \in H^{2k}(X, \mathbb{C})$ are zero.*

Next we study vector bundles which are not necessarily homogeneous.

Theorem 11. *Let $X = G/\Gamma$ be a compact complex parallelizable manifold, where G is semisimple.*

Then every line bundle is topologically equivalent to a homogeneous line bundle. Moreover $NS(X) = Pic(X)/Pic^0(X)$ is finite and therefore the first Chern class $c_1(E)$ vanishes for every holomorphic vector bundle $E \to X$.

If G has no $SL_2(\mathbb{C})$-factors, then $Pic(X)$ itself is finite.

For certain quotients $X = SL_2(\mathbb{C})/\Gamma$ we can prove the existence of non-homogeneous vector bundles.

Theorem 12. *Let Γ be a torsion-free discrete cocompact subgroup of $SL_2(\mathbb{C})$, $X = SL_2(\mathbb{C})/\Gamma$, C an elliptic curve in X. Assume that Γ/Γ' is finite.*

Then there exists a holomorphic vector bundle E of rank two with a section $\sigma \in H^0(X, E)$ such that C is the zero-set of σ.

Moreover, this vector bundle E is not homogeneous.

Recall that such a quotient manifold $SL_2(\mathbb{C})/\Gamma$ always contains elliptic curves by the results of Chapter A.

We also prove that elliptic curves are the only smooth curves in $SL_2(\mathbb{C})/\Gamma$ which can arise as zero-sets of sections in holomorphic rank two vector bundles.

Instead of vector bundles we usually will consider the associated $GL_n(\mathbb{C})$-principal bundle, since usually the relevant properties hold for a vector bundle if and only if they hold for the associated $GL_n(\mathbb{C})$-principal bundle.

Many of the above results generalize to principal bundles with a structure group other than $GL_n(\mathbb{C})$.

5. Subvarieties

Submanifolds of complex-parallelizable manifolds have been studied in [5]; in particular this article contains results on the structure of the pluricanonical fibration, about the existence of closed orbits and on bounds for the dimension of submanifolds. Furthermore, it is proved that for every holomorphic map from $\mathbb{C}$ to a compact complex-parallelizable manifold $X = G/\Gamma$ the analytic Zariski-closure of the image is an orbit of a Lie subgroup of G.

Here we want to mention some additional results concerning the cycle space and submanifold of nilmanifolds.

5a. Cycle space

For a compact Kähler space every irreducible component of the cycle space is compact. Compact complex-parallelizable manifold are never Kähler (except tori) and there are often non-compact irreducible components of the cycle space.

Theorem 13. *Let $X = G/\Gamma$ be a compact complex parallelizable manifold, $\mathcal{C}(X)$ its cycle space.*

Let Z be a compact complex submanifold of X and assume that Z is again parallelizable.

Then G acts transitively on the irreducible component of $\mathcal{C}(X)$ containing $[S]$ and this component is compact if and only if Z is the orbit of a normal Lie subgroup of G.

This can be used to deduce non-compactness results for the cycle space.

Proposition 5. *Let $X = G/\Gamma$ be a compact complex-parallelizable manifold and assume that G is not nilpotent.*

Then $\mathcal{C}(X)$ contains non-compact irreducible components.

5b. Subspaces of nilmanifolds

Nilmanifolds do not contain submanifolds transversal the Albanese map.

Theorem 14. *Let X be a complex nilmanifold, i.e. $X = G/\Gamma$ where G denotes a complex nilpotent Lie group and Γ a discrete cocompact subgroup. Let $a : X \to Alb(X)$ denote the Albanese map and $Z \subset X$ a closed analytic subset.*

Then $a(Z) = Alb(X)$ if and only if $Z = X$.

We are interested in the existence of submanifolds in parallelizable manifolds which are neither orbits nor contained in orbits.

Definition. *A submanifold Z in a complex parallelizable manifold X is called generating if the normal bundle is ample.*

In particular, if $Z \subset X = G/\Gamma$ is a generating submanifold of a complex-parallelizable manifold, then Z can not be contained in an orbit of a proper Lie subgroup of G.

Theorem 15. *Let X be a complex nilmanifold and assume that its Albanese variety $Alb(X)$ is isogenous to the Jacobian of a curve C.*

Then there exists a deformation of X which is again a complex nilmanifold and contains a one-dimensional generating submanifold.

The key ingredient for the proof is the following result on curve-lifting, which was motivated by an idea of F. Campana and H. Flenner.

Theorem 16. *Let $X \to Y$ be a surjective holomorphic map of complex nilmanifolds, $C \subset Y$ a curve for which $Alb(C) \to Alb(X)$ is a finite map.*

Then there exists a deformation X' of X as complex nilmanifold admitting a holomorphic surjection on Y such that C lifts to a curve in X'.

6. Cohomology

We proved that a holomorphic action of a complex Lie group on a complex manifold induces a holomorphic-linear action on the cohomology group of the structure sheaf $\mathcal{O}$ (this has also independently been proved by F. Lescure [6]) and used this to generalize a vanishing result of Raghunathan [12]. We obtain the following result.

Proposition 6. *Let Γ be a discrete cocompact subgroup of a semisimple complex Lie group S and assume $\Gamma/[\Gamma,\Gamma]$ is finite.*

Then $H^1(S/\Gamma, \mathcal{O}) = \{0\}$.

As a consequence such a manifold is rigid, i.e. there are no small deformations of S/Γ.

For the case $S = SL_2(\mathbb{C})$ similar and better results have been obtained independently by E. Ghys [4].

7. Automorphisms

We conclude this survey with a result on automorphisms. First we remark that it is easy to describe the connected component of the automorphism group: *If G is a complex Lie group and Γ a discrete subgroup such that every holomorphic function on $X = G/\Gamma$ is constant, then the connected component of the automorphism group is isomorphic to $G/(\Gamma \cap Z)$ where Z denotes the center of G.*

Proposition 7. *Let $X = G/\Gamma$ be a complex parallelizable manifold with G semisimple.*

Then $\mathrm{Aut}_{\mathcal{O}}(X)$ has only finitely many components.

Furthermore every surjective holomorphic self-map is an automorphism.

(This is in contrast to the case of tori. Tori always admit surjective holomorphic self-maps and their automorphism group may have infinitely many connected components.)

References

[1] A. Baker, *Transcendental Number Theory,* Cambridge University Press (1975)

[2] W. Barth, M. Otte, *Über fast-uniforme Untergruppen komplexer Liegruppen und auflösbare komplexe Mannigfaltigkeiten,* Comm. Math. Helv. **44** (1969), 269–281

[3] Y. Benoist, *Une nilvariete non affine,* J. Diff. Geo. **41** (1995), 21–52

[4] E. Ghys, *Sur les deformations de $SL_2(\mathbb{C})/\Gamma$,* (to appear).

[5] A.T. Huckleberry, J. Winkelmann, *Subvarieties of parallelizable manifolds,* Math. Ann. **295** (1993), 469–483

[6] F. Lescure, *Cohomologie totale et courants de Dolbeault invariants,* preprint (1994)

[7] A. Malcev, *On a class of homogeneous spaces,* Izvestiya Akad. Nauk SSSR Ser. Math. **13** (1949) / AMS Transl. no. **39** (1951)

[8] Y. Matsushima, *Fibrés holomorphes sur un tore complexe,* Nagoya Math. J. **14** (1959), 1–24

[9] G.A. Margulis, *Discrete Subgroups of Semisimple Lie Groups,* Springer Berlin Heidelberg New York (1989)

[10] G.D. Mostow, *Factor spaces of solvable groups,* Ann. of Math. **60** (1954), 1–27

[11] I. Nakamura, *Complex parallisable manifolds and their small deformations,* J. Diff. Geo. **10** (1975), 85–112

[12] M.S. Raghunathan, *Vanishing Theorems for Cohomology Groups Associated To Discrete Subgroups of Semisimple Lie Groups,* Osaka J. Math. **3** (1966), 243–256

[13] M.S. Raghunathan, *Discrete subgroups of Lie groups,* Erg. Math. Grenzgeb. **68** , Springer (1972)

[14] Y. Sakane, *On Compact Complex parallelisable Solvmanifolds,* Osaka J. Math. **13** (1976), 187–219

[15] G. Shimura, Y. Taniyama, *Complex multiplications of abelian varieties and its applications to number theory,* Publ. Math.Soc. Japan **6** (1961)

[16] H. Wang, *Complex parallisable manifolds,* Proc. A.M.S. **5** (1954), 771–776

[17] J. Winkelmann, *Every compact complex manifold admits a holomorphic vector bundle,* Revue Roum. Math. Pures et Appl. **38** (1993), 743–744

[18] J. Winkelmann, *Complex Analytic Geometry of Complex Parallelizable Manifolds. Habilitationsschrift,* (1994)

[19] J. Winkelmann, *Arithmeticity results for complex nilmanifolds,* (submitted for publication)