HOLOMORPHIC SELF-MAPS OF PARALLELIZABLE MANIFOLDS

J. WINKELMANN *

Mathematisches Institut
NA 4/69
Ruhr-Universität Bochum
D-44780 Bochum
Germany

jw@cplx.ruhr-uni-bochum.de

Abstract. We investigate holomorphic self-maps of complex manifolds of the form G/Γ where G is a complex Lie group and Γ a lattice. We show that they are induced by automorphisms of G and that a surjective holomorphic self-map can be nonbijective only in the directions of the nilradical of G.

Survey

In this article we investigate automorphisms and more generally self-maps of complex-parallelizable manifolds of the form $X = G/\Gamma$, where G denotes a simply-connected complex Lie group and Γ a discrete subgroup. Mostly we assume that Γ is a lattice, i.e., G/Γ is of finite volume (with respect to the Haar measure of G.)

In particular we prove that, for a lattice Γ in a semisimple complex Lie group G, every surjective holomorphic self-map f of the complex manifold $X = G/\Gamma$ with $f(e\Gamma) = e\Gamma$ is already an automorphism and of the form

$$f : g\Gamma \mapsto \phi(g)\Gamma$$

where ϕ denotes an automorphism of G (as complex Lie group) with $\phi(\Gamma) = \Gamma$. As a consequence we obtain that, for semisimple G, the automorphism group of such a complex manifold G/Γ has only finitely many connected components and its connected component is isomorphic to $G/Z \cap \Gamma$ where Z denotes the (finite) center of G.

If G is not semisimple, there may exist surjective holomorphic self-maps of G/Γ that are not bijective. However, they can be nonbijective only in the direction of the nilradical. To give a precise statement, let us assume that G is a simply-connected complex Lie group, N its nilradical, Γ a lattice in G, and $f : G/\Gamma \to G/\Gamma$ a surjective holomorphic map with $f(e\Gamma) = e\Gamma$.

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Then there exists an automorphism ϕ of G with $\phi(\Gamma) \subset \Gamma$ such that $f(g\Gamma) = \phi(g)\Gamma$ and there is a commutative diagram of holomorphic mappings

$$\begin{array}{ccc} G/\Gamma & \xrightarrow{f} & G/\Gamma \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ G/N\Gamma & \xrightarrow[F]{\sim} & G/N\Gamma \end{array}$$

with $F(gN\Gamma) = \phi(g)N\Gamma$ and $\pi(g\Gamma) = gN\Gamma$ such that F is biholo morphic.

For the case where Γ is cocompact, Theorem 1 and 2 are already contained in [12].

Notations and basic facts

In this paper complex manifolds are always assumed to be connected, Hausdorff, finite-dimensional and paracompact. We call a complex manifold *parallelizable* if it is biholomorphic to a quotient G/Γ of a complex Lie group G by a discrete subgroup Γ. There is no loss in generality in assuming G to be simply-connected (as we will usually do). For a compact complex manifold it is well-known that they are parallelizable in our sense if and only if the tangent bundle is holomorphically trivial ([11]).

A closed subgroup Γ in a Lie group G is called *cocompact* if the quotient G/Γ is compact. A discrete subgroup Γ is called a *lattice* if the quotient G/Γ admits a G-invariant probability measure. In this case we also say that G/Γ has finite volume. A discrete cocompact subgroup is necessarily a lattice, but the converse holds only for subgroups in solvable groups. (As general reference for discrete subgroups in Lie groups we mention [10],[7],[12].)

The theory of lattices in *complex* Lie groups is in some aspects easier than the theory of lattices in real Lie groups, mainly because there are no compact complex semisimple Lie groups. In particular, if G is a simply-connected complex Lie group and Γ a lattice, then Γ is "Zariski-dense" in G in the following sense: For every homomorphism ρ of complex Lie groups from G to a complex linear-algebraic group H the Zariski-closure of $\rho(G)$ and $\rho(\Gamma)$ coincide [5]. Furthermore every Γ-invariant holomorphic function on G is constant. If R denotes the radical of G and N the nilradical, then both $N/(N \cap \Gamma)$ and $R/(R \cap \Gamma)$ are compact.

Maps to parallelizable manifolds are equivariant

We start with a basic lemma on holomorphic mappings between parallelizable manifolds.

Lemma 1. *Let X be a complex manifold on which a simply-connected complex Lie group G acts. Let $Y = H/\Lambda$ be a complex-parallelizable manifold, $f : X \to Y$ a holomorphic map and assume that every holomorphic function on X is constant. Then f is equivariant with respect to a homomorphism of complex Lie groups $\phi : G \to H$.*

Proof. We have to show that the G-fundamental vector fields on X can be pushed forward to H-fundamental vector fields on Y. There are H-fundamental vector fields $v_1, \ldots, v_n$ on Y (with $n = dim_{\mathbb{C}}(Y)$) which give a global trivialization of TY. Now let w be a fundamental G-vector field on X. Using the trivialization of TY we obtain holomorphic functions g_i on X such that $f_*(w_x) = \sum_i g_i(x) v_i$ for every $x \in X$. Since every holomorphic function on X is constant, it follows that the pushed-forward tangent vectors $f_*(w_x)$ constitute a vector field $\sum_i g_i v_i$ with $g_i \in \mathbb{C}$. Thus we obtain a homomorphism of Lie algebras $f_* : \mathcal{L}ie\, G \to \mathcal{L}ie\, H$. Since G was assumed to be simply-connected, there exists an associated homomorphism of Lie groups $\phi : G \to H$. $\square$

We are now able to deduce a structure theorem for holomorphic mappings between complex parallelizable manifolds.

Theorem 1. *Let $f : X \to Y$ be a holomorphic map between complex parallelizable manifolds $X \simeq G/\Gamma$, $Y \simeq H/\Lambda$. Assume that every holomorphic function on X is constant. Then there exists a homomorphism of complex Lie groups $F : G \to H$ with $F(\Gamma) \subset \Lambda$ and an element $x \in H$ such that $f(g\Gamma) = x \cdot F(g)\Lambda$. Two such pairs (F, x), (F', x') correspond to the same map if and only if there exists an element $\lambda \in \Lambda$ such that $x' = x\lambda$ and $F'(g) = \lambda^{-1} F(g) \lambda$.*

Proof. Choose $x \in G$ such that $f(e\Gamma) = x\Lambda$. By the above lemma there exists a homomorphism of complex Lie groups $\phi : G \to H$ such that

$$f(g\Gamma) = f(g \cdot e\Gamma) = \phi(g) \cdot x\Lambda.$$

Let F be defined by $F(g) = x^{-1}\phi(g)x$. For $\gamma \in \Gamma$ we obtain

$$x\Lambda = f(e\Gamma) = f(\gamma\Gamma) = xF(\Gamma)\Lambda.$$

Hence $F(\Gamma) \subset \Lambda$.

Finally consider two maps f, f' given by $f(g\Gamma) = xF(g)\Lambda$ (resp. $f'(g\Gamma) = x'F'(g)\Lambda$). If $f = f'$, then $f(e\Gamma) = f'(e\Gamma)$ implies $x' = x\lambda$ for some $\lambda \in \Lambda$. Now

$$\zeta : g \mapsto F(g)^{-1} x^{-1} x' f'(g)$$

is a continuous map defined on the connected group G. The equality $f = f'$ implies that $\zeta(g) \in \Lambda$ for all $g \in G$. Hence ζ is constant. Since $\zeta(e) = \lambda$, it follows that $F' = \lambda^{-1} F \lambda$. $\square$

Corollary 1. *Let G be a simply-connected complex Lie group, Γ a discrete subgroup such that $X = G/\Gamma$ has no nonconstant holomorphic functions. Let $f : X \to X$ be a holomorphic self-map such that $D(f)$ has maximal rank in one point.*

Then f is surjective, $D(f)$ has maximal rank everywhere and there exists an automorphism $\phi \in Aut(G)$ with $\phi(\Gamma) \subset \Gamma$ and an element $x \in G$ such that $f(g\Gamma) = x\phi(g)\Gamma$.

Proof. Due to the theorem f is equivariant. This implies that f is surjective and $D(f)$ has maximal rank everywhere as soon as $D(f)$ has maximal rank at one point. The theorem furthermore implies that we can find a homomorphism of complex Lie groups $\phi : G \to G$ with discrete fibers which has the desired properties. However, for a simply-connected Lie group such a ϕ is necessarily an automorphism. This follows from the usual duality between simply-connected Lie groups and Lie algebras because such a ϕ evidently induces an automorphism of the Lie algebra of G. $\square$

Given a complex Lie group G and a discrete subgroup Γ, we denote the set of all holomorphic Lie group automorphisms ϕ of G with $\phi(\Gamma) = \Gamma$ by $Aut(G, \Gamma)$.

Corollary 2. *Let G be a simply-connected complex Lie group, Γ a discrete subgroup such that G/Γ has no nonconstant holomorphic functions. Let $A_0 = Aut(G, \Gamma) \times G$ be the natural semidirect product given by the natural action of $Aut(G, \Gamma)$ on G. Then $Aut(G/\Gamma) = A_0/\Gamma$ where Γ is embedded in A_0 by*

$$\gamma \mapsto (int_\gamma, \gamma)$$

(with int_γ denoting conjugation by γ).

The group of connected components of $Aut(X)$

We will now study the quotient group $Aut(X)/Aut(X)^0$, where $Aut(X)^0$ denotes the connected component of e (with respect to the compact-open topology). We will describe this quotient group and show that it is always countable and that, if G is semisimple, it is finite.*

Our previous results imply that $Aut(X)/Aut(X)^0$ is in fact isomorphic to $Aut(G, \Gamma)/Int(\Gamma)$. Hence we have to consider the size of $Aut(G, \Gamma)$.

Lemma 2. *Let G be a simply-connected complex Lie group and Γ a discrete subgroup. Assume that every holomorphic function on G/Γ is constant. Then $Aut(G, \Gamma)$ naturally embeds into $Aut(\Gamma)$ and is countable.*

Proof. Let $\phi \in Aut(G, \Gamma)$ and assume that ϕ acts trivially on Γ. Then

$$\zeta(g\Gamma) = g \cdot \phi(g^{-1})$$

defines a holomorphic map from G/Γ to G with $\zeta(e\Gamma) = e$. But G (being a simply-connected complex Lie group) is holomorphically separable. It follows that $\zeta \equiv e$ which implies $\phi = id$. Hence $Aut(G, \Gamma)$ embeds into $Aut(\Gamma)$. On the other hand $Aut(G, \Gamma)$ is a closed subgroup of the Lie group $Aut(G)$. These facts together imply that $Aut(G, \Gamma)$ is countable. $\square$

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*There are complex manifolds X for which $Aut(X)/Aut(X)^0$ is not countable. For example, let $S = \{0\} \times \mathbb{Z} \subset \mathbb{C}^2$. Then $Aut(X)/Aut(X)^0$ is uncountable for $X = \mathbb{C}^2 \setminus S$, because the whole permutation group of S embeds into $Aut(X)/Aut(X)^0$.

Corollary 3. *Let G be a simply-connected complex Lie group, Γ a discrete subgroup. Assume that every holomorphic function on G/Γ is constant. Then $Aut(X)/Aut(X)^0$ is countable.*

We will now restrict to the case where G is semisimple and prove that in this case $Aut(X)/Aut(X)^0$ is even finite. Here it is crucial that almost all automorphisms of semisimple Lie groups are inner. In fact we prove finiteness of $Aut(X)/Aut(X)^0$ for all Lie groups G with $Out(G) = Aut(G)/Int(G)$ finite. But first let us make a general remark and then proceed to an example.

The remark is the following. For a simply-connected Lie group G the automorphism group $Aut(G)$ is isomorphic to the automorphism group of its Lie algebra and thereby easily seen to be linear-algebraic. In particular, $Aut(G)$ has only finitely many connected components. Thus in order to check finiteness of $Out(G)$ it suffices to check that $Aut(G)$ and $Int(G) = G/Z$ have the same dimension.

Now let us produce an example of a nonsemisimple Lie group for which $Out(G)$ is finite and which does admit a lattice.

Example. Let $A \in SL(2,\mathbb{Z})$ with $|trace(A)| \neq 2$. Then A generates an infinite cyclic discrete subgroup Λ in $SL(2,\mathbb{C})$. Let T denote the Zariski-closure of Λ in $SL(2,\mathbb{C})$. Now T is a maximal torus of $SL(2,\mathbb{C})$ and the natural semidirect product $G_0 = T \times (\mathbb{C}^2,+)$ admits a lattice, viz. $\Lambda \times (\mathbb{Z}^2,+)$. One can easily check that $Out(G) \simeq \mathbb{Z}/2\mathbb{Z}$, where G denotes the universal covering of G_0.

Lemma 3. *Let G be a complex Lie group, Γ a lattice and assume that $Out(G) = Aut(G)/Int(G)$ is finite. Then $Aut(G,\Gamma)/Int(\Gamma)$ is finite.*

Proof. Let H denote the subgroup of $Aut(G,\Gamma)$ containing inner automorphisms of G. Then $Aut(G,\Gamma)/H \subset Out(G)$ and $H \simeq N/(N \cap Z)$, where Z denotes the center of G and N the normalizer of $\Gamma \in G$. Observe that $N^0 = Z^0$ and that N is a closed subgroup of G containing Γ. Hence N/Z^0 is a lattice in G/Z^0 which contains the lattice $\Gamma Z^0/Z^0$. It follows that $N/\Gamma Z^0$ is finite. This completes the proof of the finiteness of $Aut(G,\Gamma)/Int(\Gamma)$. □

Together, the preceding lemmata imply the following result.

Theorem 2. *Let G be a complex Lie group for which $Out(G)$ is finite (e.g. G semisimple), Γ a lattice and $X = G/\Gamma$. Then $Aut(X)/Aut(X)^0$ is finite.*

Self-maps

Here we will demonstrate that except in the directions of the nilradical, a surjective self-map of a parallelizable manifold is necessarily biholomorphic.

We start with the following observation.

Lemma 4. *Let G be a Lie group, Γ a lattice and ϕ an automorphism of G with $\phi(\Gamma) \subset \Gamma$. Then $\#\Gamma/\phi(\Gamma) = |\, det\, Ad(\phi)|$ where $Ad(\phi)$ denotes the associated automorphism of the Lie algebra $\mathcal{L}ie(G)$.*

Proof. The Haar measure on G can be realized by integrating a G-invariant volume form ω which can be considered as an element in $\wedge^n \mathcal{L}ie(G)^*$. Thus

$$\int_{G/\phi(\Gamma)} \omega = \int_{G/\Gamma} \phi^* \omega$$

and $\omega = |\, det(Ad(\phi))|\phi^*\omega$. $\square$

Corollary 4. *Let G be a semisimple Lie group, Γ a lattice and ϕ an automorphism of G with $\phi(\Gamma) \subset \Gamma$. Then $\phi(\Gamma) = \Gamma$.*

Corollary 5. *Let S be a semisimple complex Lie group, Γ a lattice and f a surjective holomorphic self-map of the complex manifold G/Γ. Then f is bijective, i.e., an automorphism.*

However, for nonsemisimple Lie groups it is possible that $\phi(\Gamma) \subsetneq \Gamma$.

Example. For a commutative ring A define

$$G_A = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} : x, y, z \in A \right\},$$

Then $\Gamma = G_{\mathbb{Z}+i\mathbb{Z}}$ is a lattice in the complex Lie group $G = G_{\mathbb{C}}$ and

$$\phi : \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & nx & n^2 z \\ & 1 & ny \\ & & 1 \end{pmatrix}$$

is an automorphism of G with $\phi(\Gamma) \subsetneq \Gamma$, if $n \in \mathbb{N}$ with $n \geq 2$.

More generally, we have the following construction.

Lemma 5. *Let g be a nilpotent Lie algebra, defined over $\mathbb{Q}$. Assume that g is graded, i.e., there are linear subspaces g_k of g (with $k \in \mathbb{N}$) such that $g = \bigoplus_k g_k$ as a vector space and $[g_m, g_n] \subset g_{m+n}$ for all m, n (again everything being defined over $\mathbb{Q}$). Then the simply-connected nilpotent complex Lie group G corresponding to $g \otimes_{\mathbb{Q}} \mathbb{C}$ admits a lattice Γ and an automorphism ϕ such that $\phi(\Gamma) \subsetneq \Gamma$ (provided $g_0 \subsetneq g$).*

Proof. For any positive natural number N we can define an automorphism Φ_N of g if we set $\Phi_N(v) = N^k v$ for $v \in g_k$ and extend linearly to the whole of g. The grading being defined over $\mathbb{Q}$ allows us to find a $\mathbb{Z}$-module Λ inside g such that $\Lambda \otimes_Z \mathbb{Q} \simeq g$ and $\Lambda = \bigoplus_k (g_k \cap \Lambda)$. Thus $\Phi_N(\Lambda) \subset \Lambda$. By a result of Malcev [6] $exp(\Lambda + i\Lambda)$ generates a lattice Γ in G. Evidently $\phi(\Gamma) \subset \Gamma$ if ϕ is an automorphism of G corresponding to Φ_N for some N.

On the other hand $|\,det\,\Phi_N| > 1$ whenever $N > 1$ and $g \neq g_0$. Therefore $\phi(\Gamma) \neq \Gamma$. □

Remark. There are nilpotent $\mathbb{Q}$-Lie algebras which do not admit any non-trivial grading [4].

We prove that for arbitrary (not necessarily nilpotent) complex Lie groups the inclusion $\phi(\Gamma) \subset \Gamma$ can be nonbijective only in the directions of the nilradical.

We begin with a lemma on lattice-preserving automorphisms of nilpotent Lie groups.

Lemma 6. *Let N be a simply-connected real nilpotent Lie group, Γ a lattice, and ϕ an automorphism of N with $\phi(\Gamma) \subseteq \Gamma$. Let $\chi_\phi(t)$ denote the characteristic polynomial of the induced automorphism of $\mathcal{L}ie(N)$. Then $\chi_\phi \in \mathbb{Z}[t]$.*

Proof. First let us consider the case where N is abelian. In this case $N \simeq \mathbb{R}^n$ and $\Gamma \simeq \mathbb{Z}^n$ in suitable coordinates. Hence $\phi \in M(n, \mathbb{Z})$ and evidently $\chi_\phi \in \mathbb{Z}[t]$.

Now the general proof can be deduced by induction. Induction is justified by Malcev's result [6] that for every lattice Γ in N and every subgroup of the central series N^k the intersection $N^k \cap \Gamma$ is a lattice in N^k. □

Proposition 1. *Let G be a simply-connected solvable real Lie group, Γ_k an sequence of lattices in G. Let I^0 denote the connected component of the closure of the union $\cup_k \Gamma_k$. Then I^0 is contained in the nilradical N of G.*

Proof. For $g \in G$ let $Ad(g)$ denote the adjoint automorphism of $\mathcal{L}ie(G)$ and denote the characteristic polynomial of $Ad(g)|_N$ by χ_g. Now $N \cap G$ is a lattice in G ([9], Lemma 3.9). Thus for every k and every $\gamma \in \Gamma_k$ the characteristic polynomial χ_γ of $Ad(\gamma)|_N$ is contained in $\mathbb{Z}[x]$. By continuity this implies that for every $g \in I^0$ the characteristic polynomial of $Ad(g)|_N$ is trivial, i.e., $Ad(g)|_N$ is unipotent. Since N contains the commutator group of G, $Ad(g)_{\mathcal{L}ie\,N}$ is nilpotent if and only if $g \in N$. It follows that I^0 is contained in the nilradical. □

Lemma 7. *Let G denote a Lie group, Γ a lattice, ϕ an automorphism of G with $\phi(\Gamma) \subset \Gamma$. Then either $\phi(\Gamma) = \Gamma$ or $\hat{\Gamma} = \cup_{k \geq 0} \phi^{-k}(\Gamma)$ is a nondiscrete subgroup of G.*

Proof. Since Γ is a lattice, it is clear that either $\hat{\Gamma}$ is a lattice or $\hat{\Gamma}$ is not discrete. Let $d = [\phi(\Gamma) : \Gamma] = \#(\Gamma/\phi(\Gamma))$. Then $vol(G/\phi^{-k}(\Gamma)) = d^{-k} \cdot vol(G/\Gamma)$. Thus $\hat{\Gamma}$ can be discrete only in the case where $lim_k\, d^{-k} > 0$, i.e., only if $d = 1$. □

Corollary 6. *Let G be a simply-connected complex solvable Lie group, N its nilradical, Γ a lattice in G and ϕ an automorphism of G with $\phi(\Gamma) \subset \Gamma$. Then $\phi(\Gamma)N = \Gamma N$.*

Proof. First note that $\cup_k \phi^{-k}(\Gamma)$ is a subgroup of G because it is the union of an ascending sequence of subgroups. Let I denote the closure of $\cup_k \phi^{-k}(\Gamma)$ in G and denote its connected component by I^0. Evidently I^0 is normalized by Γ. Now $Ad(G)$ and $Ad(\Gamma)$ have the same Zariski-closure in $Aut(\mathcal{L}ie\, G)$. Thus $Ad(G)$ must stabilize $\mathcal{L}ie\, I$ and consequently I^0 is a normal Lie subgroup of G.

We will now consider the quotient Lie group G/I^0. Since $\Gamma \subset I$, the quotient G/I admits an invariant finite measure ([8]), hence I/I^0 is a lattice in G/I^0. On the other hand $I^0\Gamma$ is also of finite covolume and therefore $\Gamma_0 = \Gamma/(\Gamma \cap I^0)$ is likewise a lattice. It follows that Γ_0 is a lattice in G/I^0 with $\cup_k \phi_0^{-k}(\Gamma_0)$ being discrete (where ϕ_0 is the automorphism of G/I^0 induced by ϕ.) Thanks to the preceding lemma we may now conclude $\phi_0(\Gamma_0) = \Gamma_0$, i.e., $\phi(\Gamma)I^0 = \Gamma I^0$. This completes the proof since $I^0 \subset N$. $\square$

We are now in a position to prove the following theorem.

Theorem 3. *Let $X = G/\Gamma$ be a complex-parallelizable manifold with finite volume. Let f be a surjective holomorphic self-map. Let N denote the nilradical of G. Then there is a commutative diagram*

$$\begin{array}{ccc} G/\Gamma & \xrightarrow{f} & G/\Gamma \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ G/N\Gamma & \underset{F}{\xrightarrow{\sim}} & G/N\Gamma, \end{array}$$

where F is an automorphism of the complex manifold $G/N\Gamma$.

Proof. There is an automorphism ϕ of the complex Lie group G with $\phi(\Gamma) \subset \Gamma$ and an element $a \in G$ such that f is given by

$$f : g\Gamma \mapsto a\phi(g)\Gamma.$$

We have to show that $\phi(\Gamma)N = \Gamma N$. Note that both the radical R and the nilradical N of G have closed orbits in G/Γ. Thus $R\Gamma/R$ is a lattice in the semisimple group G/R, implying $\phi(R\Gamma) = R\Gamma$. On the other hand $R \cap \Gamma$ is a lattice in R. Hence $\phi((R \cap \Gamma)N) = (R \cap \Gamma)N$. Together these facts yield $\phi(N\Gamma) = N\Gamma$ as desired. $\square$

Meromorphic maps

All results on holomorphic maps in this paper are also valid for meromorphic maps for the simple reason that every meromorphic map to a complex parallelizable manifold *is* already holomorphic.

Indeed, let X be a complex manifold, $Y = G/\Gamma$ a complex parallelizable manifold and let $f : X \to Y$ be a meromorphic map. Then f is holomorphic outside an analytic set E of codimension two. Now, for any simply-connected open subset $U \subset X$, $U \setminus E$ is still simply-connected. It follows that $f|_{U\setminus E}$ lifts to a holomorphic map to the universal covering G of X. Since G is Stein, this lift can be realized by holomorphic functions. By Hartog's theorem it then follows that f was holomorphic on U in the first place.

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