A Mergelyan Theorem for Mappings to $\mathbb{C}^2 \setminus \mathbb{R}^2$

by

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The classical Mergelyan theorem states that a holomorphic function on a compact set K in $\mathbb{C}$ can be approximated by global holomorphic functions provided that the complement K^c is connected. We are interested in such approximation phenomena for holomorphic mappings with values in complex manifolds. Of course the target manifold can not be arbitrary. E.g. Liouville's theorem excludes bounded domains as possible target manifolds. More generally it is easy to see that a Mergelyan theorem can not hold unless the Kobayashi pseudodistance on the target manifold vanishes identically.

Of course the Mergelyan theorem holds for maps into $\mathbb{C}^n$. Furthermore if the Hausdorff dimension of a closed set A in $\mathbb{C}^n$ is less than $2n-2$, then the theorem remains true for maps into $\mathbb{C}^n \setminus A$ by a density argument (see prop. 2). However this density argument breaks down in the case where A is a submanifold of real dimension $2n-2$. Thus we have to use different methods to prove a Mergelyan theorem in the case $codim_{\mathbb{R}}(A) = 2$.

Theorem. *Let K be a compact subset of $\mathbb{C}$ with $\mathbb{C} \setminus K$ connected, $\epsilon > 0$ and $f : K \to \mathbb{C}^2 \setminus \mathbb{R}^2$ a continuous map which is holomorphic in the interior of K.*

Then there exists a holomorphic map $g : \mathbb{C} \to \mathbb{C}^2 \setminus \mathbb{R}^2$ with $||f-g||_K < \epsilon$.

Remarks.

i) The only property of $\mathbb{C}^2 \setminus \mathbb{R}^2$ we will use is the following: $\mathbb{C}^2 \setminus \mathbb{R}^2 = \mathbb{C}^2 \setminus (A_1 \times A_2)$ with $A_i \subset \mathbb{C}$ where the sets A_i are real–analytic Weierstraß sets (e.g. $A_i = \{h_i = 0\}$ where h_i are real harmonic functions).

ii) If A is a closed subset in $\mathbb{C}^n$ such that the Hausdorff dimension of A is less than $2n-2$, then $Hol(\mathbb{C}, \mathbb{C}^n \setminus A)$ is dense in $Hol(\mathbb{C}, \mathbb{C}^n)$ (see prop. 2). Consequently a Mergelyan theorem holds for maps to $\mathbb{C}^n \setminus A$.

iii) If A is a real submanifold of real dimension $2n-2$ in $\mathbb{C}^n$, then $Hol(\mathbb{C}, \mathbb{C}^n \setminus A)$ is not dense in $Hol(\mathbb{C}^n)$ (see prop. 3).

iv) If A is a complex hypersurface in $\mathbb{C}^n$ of "sufficiently high degree", e.g. a union of $n+1$ hyperplanes in general position, then a Mergelyan theorem for mappings to $\mathbb{C}^n \setminus A$ can not hold, because $\mathbb{C}^n \setminus A$ has non–trivial Kobayashi–pseudodistance.

v) Königsberger [4] has shown that a Mergelyan theorem holds for mappings to complex Lie groups. This result has been generalized by Dietmair [2] for mappings to complex manifolds on which a complex Lie group acts transitively.

Before beginning the proof, let us fix some notation: If A is a closed set then $\mathcal{A}(A)$ (resp. $\mathcal{A}(A, Y)$ where Y is a complex manifold) denotes the set of all continuous functions (resp. maps to Y) which are holomorphic in the interior of A. A^c denotes the complement of A.

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* Part of this work was carried out during a visit at the mathematical institute of the University of Lille. The author wishes to thank the colleagues there for their hospitality.

1991 AMS Mathematics Subject Classification 30E10, 32E30, 32H99

Our first step in the proof is to reduce the problem from arbitrary compact sets to closure of domains with smooth boundary.

Lemma 1. *Let K be a compact subset of $\mathbb{C}$ with K^c connected, and U an open subset of $\mathbb{C}$ containing K.*

Then there exists an open domain G in $\mathbb{C}$ with smooth boundary ∂G such that $K \subset G \subset\subset U$. Furthermore $\bar{G}^c = \mathbb{C} \setminus \bar{G}$ is connected.

Proof. For any $x \in K^c$ and any real curve γ linking x to ∞ inside K^c define $\Omega(x,\gamma) = \inf_t(d(\gamma(t),K))$. For $x \in K^c$ let $\omega(x)$ be the supremum of $\Omega(x,\gamma)$ over all such curves γ. Define $\omega(x) = 0$ for $x \in K$. It is easy to check that the function ω has the following properties:

1) $\omega(x) = 0$ if and only if $x \in K$;
2) $|\omega(x-a) - \omega(x)| \leq |a|$ for all $x, a \in \mathbb{C}$ (in particular ω is continuous);
3) $\omega(x) = d(x,K)$ for all x which are not contained in the linear-convex hull of K. (Therefore the set $\{\omega \leq c\}$ is compact for all c).
4) The sets $\{\omega(x) > c\}$ are connected for all c.

Let $\nu = \min\{\omega(x) : x \notin U\}$. (The above listed properties of ω ensure that this minimum exists and is non–zero.) Now let ρ be a convolution of ω with a C^∞ smoothing kernel χ. We require that $supp(\chi)$ is contained in a ball of radius $\frac{1}{4}\nu$ around the origin. Due to property 2 this implies $|\omega(x) - \rho(x)| \leq \frac{1}{4}\nu$ for all x. The theorem of Sard implies that there exists a δ with $\frac{1}{4}\nu < \delta < \frac{1}{2}\nu$ such that $\rho'(x) \neq 0$ for all x with $\rho(x) = \delta$. Now let $G_0 = \{\rho < \delta\}$ and G be the union of G_0 with all bounded connected components of $\mathbb{C} \setminus G_0$. Then $K \subset G_0$ and $\omega(x) < \frac{3}{4}\nu$ for all $x \in G_0$. Since $\{\omega > \frac{3}{4}\nu\}$ is unbounded and connected, it follows that $\omega(x) \leq \frac{3}{4}\nu$ for all $x \in \bar{G}$. Hence $\bar{G} \subset U$. □

Corollary. *Let K be a compact subset of $\mathbb{C}$ with K^c connected, $f \in \mathcal{A}(\mathcal{K}, \mathbb{C}^{\in} \setminus \mathbb{R}^{\in})$ and $\epsilon > 0$.*

Then there exists an open domain G in $\mathbb{C}$ with smooth boundary ∂G and $g \in \mathcal{A}(\bar{\mathcal{G}}, \mathbb{C}^{\in} \setminus \mathbb{R}^{\in})$ such that $K \subset G$ and $||f - g||_K < \epsilon$. Furthermore $\bar{G}^c = \mathbb{C} \setminus \bar{G}$ is connected.

Proof. Let $g \in Hol(\mathbb{C}, \mathbb{C}^2)$ with $||f - g||_K < \epsilon$. Now apply the lemma with $U = g^{-1}(\mathbb{C}^2 \setminus \mathbb{R}^2)$. □

Lemma 2. *Let G be a bounded open domain in $\mathbb{C}$ with smooth boundary, $\epsilon > 0$ and f a holomorphic function on $\mathbb{C}$.*

Then there exists a holomorphic function f_1 on $\mathbb{C}$ such that $||f - f_1||_{\bar{G}} < \epsilon$ and $A = f_1^{-1}(\mathbb{R})$ intersects ∂G transversally at any point of intersection.

Proof. Let Θ be a (real differentiable) vector field on ∂G which is nowhere vanishing. Define a map $\Phi : \mathbb{R}^3 \times \partial K \to \mathbb{R}^2$ by $(a,b,c,x) \mapsto (\Theta \Re f_{a,b,c}(x), \Re f_{a,b,c}(x))$ with $f_{a,b,c}(x) = f(x) + (a+ib)z + c$. Φ has everywhere maximal rank, hence $\Phi^{-1}(0,0)$ is a two-dimensional submanifold whose projection on $\mathbb{R}^3$ is a set of measure zero. Thus for appropriate a, b, c the function $f_1 = f_{a,b,c}$ has the required properties. □

Lemma 3. *Let G be a bounded domain with smooth boundary and a defining function ρ. Assume that $\{\rho \leq a\}$ is compact for all a.*

Then there exists a positive number $c > 0$ such that for all ϵ with $0 < \epsilon < c$ the domain $G_\epsilon = \{\rho < \epsilon\}$ has smooth boundary.

Proof. Let C be the set of critical points of ρ. C is a closed set. Since $\{\rho \leq a\}$ is compact for all a, it follows that $\rho(C)$ is a closed set. □

Next we need some auxiliary results on Weierstrass sets. A closed subset A of $\mathbb{C}$ is called a Weierstrass set if any function $f \in \mathcal{A}(A)$ can be approximated uniformly on A by global holomorphic functions. The theorem of Arakelyan [1] states that a closed subset $A \subset \mathbb{C}$ is a Weierstrass set if and only if the following two conditions are fulfilled:

i) $\bar{\mathbb{C}} \setminus A$ is connected and

ii) $\bar{\mathbb{C}} \setminus A$ is locally connected at ∞.

Here $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq \mathbb{P}_1$ is the one–point compactification of $\mathbb{C}$.

The latter condition is equivalent to each of the following two conditions:

ii') For every open neighbourhood U of ∞ in $\mathbb{C} \setminus A$ there exists a smaller open neighbourhood V of ∞ such that for all points $x \in V$ there exists a real curve inside U linking x with ∞.

ii") For every compact subset K of $\mathbb{C}$ the union B of all bounded connected components of $\mathbb{C} \setminus (A \cup K)$ is bounded.

Furthermore the criterion of Eilenberg guarantees that condition i) is fulfilled if A is a simply–connected and locally pathwise–connected closed set.

Lemma 4. *The set $A = f^{-1}(\mathbb{R})$ is simply–connected for all holomorphic maps $f : \mathbb{C} \to \mathbb{C}$.*

Proof. Let G be an arbitrary bounded open domain. The maximum principle implies that ∂G is not contained in A unless f is constant. From this it follows that A is simply–connected. □

Lemma 5. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic map. Then the complement A^c of $A = f^{-1}(\mathbb{R})$ is locally connected at infinity.*

Proof. Due to the preceding lemma A^c has no bounded connected components. Hence for any compact set K every bounded connected component of $\mathbb{C} \setminus (K \cup A)$ meets K. Thus in order to prove the boundedness of the union of all such components, it suffices to show that only finitely many connected components of $\mathbb{C} \setminus (K \cup A)$ meet K. By enlarging K we may assume that the boundary ∂K is real–analytic. Again by the maximum principle $\Im(f)$ is not constant on ∂K. The identity principle for real–analytic functions implies that $\Im(f)$ has only finitely many zeroes. This finishes the proof. □

Corollary. *For every holomorphic function f the set of real values $f^{-1}(\mathbb{R})$ is a Weierstrass set.*

Remark: For the above corollary it is essential that $\mathbb{R}$ is a real–analytic Weierstrass set. In general the preimage of an arbitrary Weierstrass set under a holomorphic map is not a Weierstrass set. For instance let $H : \mathbb{R}_0^+ \to \mathbb{R}^2 \simeq \mathbb{C}$ be given by $t \mapsto \left(\frac{-t}{t+1} sin^2 t + 1, e^{-t}\right)$ and let A be the union of $[0,1] \times \{0\}$ with the image of H. Then A is a closed Weierstrass set, but $exp^{-1}(A)$ is not locally connected at infinity.

Proposition 1. *Let G be a bounded domain with smooth boundary and G^c connected, $f \in \mathcal{A}(\bar{G})$, and $\epsilon > 0$.*

Then there exists a holomorphic function $f \in \mathcal{O}(\mathbb{C})$ such that $||f - f_1||_{\bar{G}} < \epsilon$ and $\bar{G} \cup A$ is a Weierstrass set for $A = \{\Re f_1 = 0\}$. Furthermore one can require that each connected component of $A \setminus G$ intersects ∂G in at most one point.

Proof. Let ρ be a defining function such that G_δ^c is connected for all $0 < \delta$. (This is possible, because there is a diffeomorphism of $\mathbb{C}$ mapping G onto the unit disk.) Due to the Mergelyan theorem and Lemma 2 there is a function $f_1 \in \mathcal{O}(\mathbb{C})$ with $||f - f_1||_G < \frac{1}{2}\epsilon$ such that $A = \Im f_1 = 0$ intersects ∂G tranversally. Let $\delta_0 > 0$ be such that A intersects $\{\rho = c\}$ tranversally for all $0 \le c \le \delta_0$. Let $p_1, \ldots, p_m \in S = \{\rho = \delta_0\}$ be choosen such that the connected components of $S \setminus \{p_1, \ldots, p_m\}$ contain at most one point x with $\Im f_1(x) = 0$ each. Now let γ_i be real disjoint curves in $G_{\delta_0}^c$ linking p_i with ∞. We may assume that in a small neighbourhood of ∞ the image of each γ_i equals $\{re^{i\theta} : \theta = \theta_i\}$ for some θ_i. This implies that $\mathbb{C} \setminus L$ is locally connected at infinity, where L is the union of G_{δ_0} with the images of the γ_i. Since L is also simply–connected, it follows that L is a Weierstrass set. Let $F(x)$ defined by $F(x) = f_1(x)$ for $x \in G_{\delta_0}$ and $F(x) = f_1(p_i)$ if $x \in \gamma_i$. Let g_n be a sequence of holomorphic functions on $\mathbb{C}$ which converge uniformly on L to F. Uniform convergence of holomorphic functions implies the uniform convergence of their derivatives. Since F is holomorphic in the interior of L, it follows that for a sufficiently large n the set $A_n = \{\Im g_n = 0\}$ intersects ∂G transversally and in as many points as $\{\Im f_1 = 0\}$. Furthermore for n sufficiently large A_n has empty intersection with all γ_i. From this we can deduce that each connected component of $A_n \setminus G$ has at most one point of intersection with ∂G. This together with Lemma 4. in turn implies that $A_n \cup \bar{G}$ is simply–connected. Finally note that $A \cup \bar{G}$ is locally connected at infinity due to Lemma 5. □

Proof of the theorem. By Lemma 1 we may restrict our considerations to the case where $K = \bar{G}$ for a bounded open domain G with smooth boundary. By the classical Mergelyan theorem there is a holomorphic map $F : \mathbb{C} \to \mathbb{C}^2$ with $||f - F||_{\bar{G}} < \frac{1}{2}\epsilon$ and $||f - F||_{\bar{G}} < d(f(\bar{G}), \mathbb{R}^2)$; the latter inequality implying $F(\bar{G}) \subset \mathbb{C}^2 \setminus \mathbb{R}^2$. Let $A = \{\Im F_1 = 0\} \setminus G$, A_i the connected components of A and L the union of $\bar{G}$ with A. By the preceding proposition we may assume that L is a Weierstrass set and that each A_i intersects ∂G in at most one point. We define a function $h \in \mathcal{A}(L)$ as follows:

$$h(x) = \begin{cases} 0 & \text{if } x \in \bar{G} \\ 1 - \Im F_2(x) & \text{if } x \in A_i \text{ and } A_i \cap \partial G = \emptyset \\ -\Im F_2(x) + \Im F_2(q_i) & \text{if } x \in A_i \text{ and } A_i \cap \partial G = \{q_i\} \end{cases}$$

Now we choose $\delta > 0$ such that $\delta < d(F(\bar{G}), \mathbb{C}^2 \setminus \mathbb{R}^2)$ and $\delta < |\Im F_2(q)|$ for all $q \in A \cap \partial G$ and $\delta < \min(\frac{1}{2}\epsilon, 1)$. Let $H \in \mathcal{O}(\mathbb{C})$ with $||H - h||_L < \delta$. Then $g = (F_1, F_2 + iH)$ maps $\mathbb{C}$ to $\mathbb{C}^2 \setminus \mathbb{R}^2$ with $||f - g||_{\bar{G}} < \epsilon$. □

Complements of small sets

Assume that a Mergelyan theorem holds for maps to a given complex manifold X. Let A be a closed subset of X which is small enough to ensure that $Hol(\mathbb{C}, X \setminus A)$ is dense in $Hol(\mathbb{C}, X)$. Then obviously there is also a Mergelyan theorem for maps with values in $X \setminus A$. Therefore we will now prove such a density result.

Proposition 2. *Let X be a complex manifold on which a real Lie group G acts transitively by biholomorphic transformations.*

Let A be a closed subset of X. Assume that the Hausdorff dimension of A is less than $2n - 2$, where n denotes the complex dimension of X. Then $Hol(\mathbb{C}, X \setminus A)$ is dense in $Hol(\mathbb{C}, X)$.

Proof. For $f \in Hol(\mathbb{C}, X)$ and $g \in G$ define $f_g(x) = g \cdot f(x)$. Define $\Phi : G \times \mathbb{C} \to X$ by $\Phi(g, x) = f_g(x)$. Now Φ is surjective and has everywhere maximal rank. Hence the Hausdorff codimension of $\Phi^{-1}(A)$ equals the codimension of A, i.e. it is larger than two. Let $p : G \times \mathbb{C} \to G$ denote the natural projection. Then the Hausdorff dimension of $p(\Phi^{-1}(A))$ is smaller than $dim(G)$ which implies that the complement is dense. Hence for any $f \in Hol(\mathbb{C}, X)$ there are arbitrarily small $g \in G$ with $f_g(\mathbb{C}) \subset X \setminus A$. $\square$

Remark. The homogeneity assumption is quite essential, e.g. consider $X = \{(z, w) : |zw| < 1, |z| < 1\}$ and $A = \{(0, 0), (0, 1)\}$.

However the Mergelyan theorem for $\mathbb{C}^2 \setminus \mathbb{R}^2$ can not be deduced in this way.

Proposition 3. *Let A be a closed subset in $\mathbb{C}^n$. Assume that there exists a point $a \in A$ with an open neighbourhood U in $\mathbb{C}^n$ such that $U \cap A$ is a real submanifold of real codimension two.*

Then $Hol(\mathbb{C}, \mathbb{C}^n \setminus A)$ is not dense in $Hol(\mathbb{C}, \mathbb{C}^n)$.

Proof. We may replace U by a smaller open subset. Hence we may assume that $\pi_1(U \setminus A) \simeq \mathbb{Z}$. Furthermore there exists a complex–linear map $\lambda : \mathbb{C} \to \mathbb{C}^n$ with $\lambda(0) = a$ and $\lambda(\bar{\Delta}_1) \subset U$ such that $\lambda_* : \pi_1(\partial\Delta_1) \to \pi_1(U \setminus A)$ is an isomorphism. If a sequence $f_n \in Hol(\mathbb{C}, \mathbb{C}^n \setminus A)$ converges to λ, then $f_n(\bar{\Delta}_1) \subset U$ for almost all n. Thus $f_n(\bar{\Delta}_1) \subset U \setminus A$ for almost all n. This however implies that $f_n(\partial\Delta_1)$ is contractible inside $U \setminus A$ which leads to a contradiction. $\square$

Remark. For the proof we only need the existence of a holomorphic map λ with prescribed values $\lambda(0)$ and $d\lambda(0)$.

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