Robust A Posteriori Error Estimates for Non-Stationary Convection-Diffusion Problems

R. Verfürth

Fakultät für Mathematik
Ruhr-Universität Bochum

www.ruhr-uni-bochum.de/num1

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Goal

Establish residual a posteriori error estimates for SUPG-discretizations of non-stationary convection-diffusion problems which yield upper and lower bounds for the energy norm of the error that are uniform with respect to all possible relative sizes of convection to diffusion.

Outline

Differential Equation

$$
\begin{aligned}
\partial_t u - \operatorname{div}(d\nabla u) + \mathbf{a}\cdot\nabla u + ru &= f && \text{in } \Omega\times(0,T] \\
u &= 0 && \text{on } \Gamma\times(0,T] \\
u &= u_0 && \text{in } \Omega
\end{aligned}
$$

- $d > 0$
- $r \geq 0$
- $\mathbf{a} \in C^1(\Omega\times(0,T])^d$
- $\operatorname{div}\mathbf{a} = 0$ in $\Omega\times(0,T]$

Norms

- Energy norm

$$|||v||| = \left\{ d\|\nabla v\|^2 + r\|v\|^2 \right\}^{\frac{1}{2}}$$

- Dual norm

$$|||\varphi|||_* = \sup_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\langle \varphi, v\rangle}{|||v|||}$$

- Error norm

$$\|u\|_{X(a,b)} = \Big\{ \operatorname*{ess.\,sup}_{t\in(a,b)} \|u(\cdot,t)\|^2 + \int_a^b |||u(\cdot,t)|||^2 dt + \int_a^b |||(\partial_t u + \mathbf{a}\cdot\nabla u)(\cdot,t)|||_*^2 dt \Big\}^{\frac{1}{2}}$$

Meshes and Spaces

- $\mathcal{I} = \{(t_{n-1}, t_n] : 1 \le n \le N_\mathcal{I}\}$ partition of $[0, T]$.
- $\tau_n = t_n - t_{n-1}$.
- $\mathcal{T}_n$, $0 \le n \le N_\mathcal{I}$, affine equivalent, admissible, shape regular partitions of Ω.
- Transition condition: There is a common refinement $\widetilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ such that $h_K \le c h_{K'}$ for all $K \in \mathcal{T}_n$ and all $K' \in \widetilde{\mathcal{T}}_n$ with $K' \subset K$.
- $V_n \subset H_0^1(\Omega)$ finite element space corresponding to $\mathcal{T}_n$.

Discrete Problem

Find $u_{\mathcal{T}_n}^n \in X_n$, $0 \le n \le N_\mathcal{I}$, such that $u_{\mathcal{T}_0}^0 = \pi_0 u_0$ and, for $n = 1, \ldots, N_\mathcal{I}$ and all $v_{\mathcal{T}_n} \in X_n$

$$\int_\Omega \frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} v_{\mathcal{T}_n} + a(\theta\nabla u_{\mathcal{T}_n}^n + (1-\theta)\nabla u_{\mathcal{T}_{n-1}}^{n-1}, v_{\mathcal{T}_n})$$

$$+ \sum_{K\in\widetilde{\mathcal{T}}_n} \delta_K \int_K \Big(\frac{u_{\mathcal{T}_n}^n - u_{\mathcal{T}_{n-1}}^{n-1}}{\tau_n} + L(\theta u_{\mathcal{T}_n}^n + (1-\theta) u_{\mathcal{T}_{n-1}}^{n-1}) \Big) \mathbf{a}\cdot\nabla v_{\mathcal{T}_n}$$

$$= \int_\Omega f v_{\mathcal{T}_n} + \sum_{K\in\widetilde{\mathcal{T}}_n} \delta_K \int_K f \mathbf{a}\cdot\nabla v_{\mathcal{T}_n}$$

with

$a(u, v) = d(\nabla u, \nabla v) + (\mathbf{a}\cdot\nabla u, v) + r(u, v)$,

$Lv = -\operatorname{div}(d\nabla u) + \mathbf{a}\cdot\nabla u + ru$

Basic Steps

- Error and residual are equivalent.
- The residual splits into a spatial and a temporal residual.
- The norm of the sum of these is equivalent to the sum of their norms.
- Derive a reliable, efficient and robust error indicator for the spatial residual.
- Derive a reliable, efficient and robust error indicator for the temporal residual.

Equivalence of Error and Residual

- $u_{\mathcal{I}}$ continuous piece-wise affine, equals $u^n_{\mathcal{T}_n}$ at t_n.
- Residual:

$$\begin{aligned}\langle R(u_{\mathcal{I}}), v\rangle &= (f, v) - (\partial_t u_{\mathcal{I}}, v) - (d\nabla u_{\mathcal{I}}, \nabla v)\\ &\quad - (\mathbf{a}\cdot\nabla u_{\mathcal{I}}, v) - (r u_{\mathcal{I}}, v)\end{aligned}$$

- Lower bound:

$$\|R(u_{\mathcal{I}})\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \le \sqrt{2}\|u - u_{\mathcal{I}}\|_{X(t_{n-1},t_n)}$$

- Upper bound:

$$\|u - u_{\mathcal{I}}\|_{X(0,t_n)} \le \left\{4\|u_0 - \pi_0 u_0\|^2 + 6\|R(u_{\mathcal{I}})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}\right\}^{\frac{1}{2}}$$

Proof of the Equivalence

- Relation of residual and error:

$$\langle R(u_{\mathcal{I}}), v\rangle = (\partial_t e, v) - (\mathbf{a}\cdot\nabla e, v) - (d\nabla e, \nabla v) - (re, v)$$

- Lower bound: Definition of primal and dual norm plus Cauchy-Schwarz inequality.
- Upper bound: Parabolic energy estimate with $v = e$ as test-function.

Decomposition of the Residual

- Temporal residual:

$$\begin{aligned}\langle R_\tau(u_{\mathcal{I}}), v\rangle &= (d\nabla(u^n_{\mathcal{T}_n} - u_{\mathcal{I}}), \nabla v) + (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u_{\mathcal{I}}), v)\\ &\quad + (r(u^n_{\mathcal{T}_n} - u_{\mathcal{I}}), v)\end{aligned}$$

- Spatial residual:

$$\begin{aligned}\langle R_h(u_{\mathcal{I}}), v\rangle &= (f, v) - (\partial_t u_{\mathcal{I}}, v) - (d\nabla u^n_{\mathcal{T}_n}, \nabla v)\\ &\quad - (\mathbf{a}\cdot\nabla u^n_{\mathcal{T}_n}, v) - (r u^n_{\mathcal{T}_n}, v)\end{aligned}$$

- Splitting: $R(u_{\mathcal{I}}) = R_\tau(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}})$
- Estimate for $L^2(t_{n-1}, t_n; H^{-1}(\Omega))$-norms:

$$\begin{aligned}\frac{1}{5}\left\{\|R_\tau(u_{\mathcal{I}})\|^2 + \|R_h(u_{\mathcal{I}})\|^2\right\}^{\frac{1}{2}} &\le \|R_\tau(u_{\mathcal{I}}) + R_h(u_{\mathcal{I}})\|\\ &\le \|R_\tau(u_{\mathcal{I}})\| + \|R_h(u_{\mathcal{I}})\|\end{aligned}$$

Motivation of the Lower Bound

- Strengthened Cauchy-Schwarz inequality for $v = c$ and $w = \frac{b-t}{b-a}$:

$$\int_a^b vw = \frac{1}{2}c(b-a) = \frac{\sqrt{3}}{2}\|v\|_{(a,b)}\|w\|_{(a,b)}$$

- Hence:

$$\|v + w\|^2_{(a,b)} \ge \left(1 - \frac{\sqrt{3}}{2}\right)\left\{\|v\|^2_{(a,b)} + \|w\|^2_{(a,b)}\right\}$$

Proof of the Lower Bound

- $R_h(u_{\mathcal{I}})$ is piece-wise constant.
- $R_\tau(u_{\mathcal{I}})$ is piece-wise affine: $R_\tau(u_{\mathcal{I}}) = \frac{t_n - t}{\tau_n}\rho^n$ with

$$\langle \rho^n, v\rangle = (d\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), \nabla v) + (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v) + (r(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v).$$

- Choose $v, w \in H^1_0(\Omega)$ such that

$$|||v||| = |||R_h(u_{\mathcal{I}})|||_*, \qquad \langle R_h(u_{\mathcal{I}}), v\rangle = |||R_h(u_{\mathcal{I}})|||^2_*,$$
$$|||w||| = |||\rho^n|||_*, \qquad \langle \rho^n, w\rangle = |||\rho^n|||^2_*.$$

- Insert $3\left(\frac{t - t_{n-1}}{\tau_n}\right)^2 v + \frac{t_n - t}{\tau_n} w$ as test-function in representation of $R(u_{\mathcal{I}})$.

Estimation of the Spatial Residual

- Spatial error indicator $\eta^n_{\mathcal{T}_n}$:

$$\eta^n_{\mathcal{T}_n} = \Big\{ \sum_{K \in \widetilde{\mathcal{T}}_n} \alpha_K^2 \|R_K\|_K^2 + \sum_{E \in \mathcal{E}_{\widetilde{\mathcal{T}}_n}} \varepsilon^{-\frac{1}{2}} \alpha_E \|R_E\|_E^2 \Big\}^{\frac{1}{2}}$$

- $\alpha_S = \min\{d^{-\frac{1}{2}} h_S, r^{-\frac{1}{2}}\}$
- R_K and R_E are the usual element and interface residuals.
- Standard arguments for stationary problems yield:

$$|||R_h(u_{\mathcal{I}})|||_* \le c^\dagger \eta^n_{\mathcal{T}_n},$$
$$\eta^n_{\mathcal{T}_n} \le c_\dagger |||R_h(u_{\mathcal{I}})|||_*.$$

- $c^\dagger c_\dagger$ only depend on the polynomial degrees and on the shape parameters of the partitions $\widetilde{\mathcal{T}}_n$.

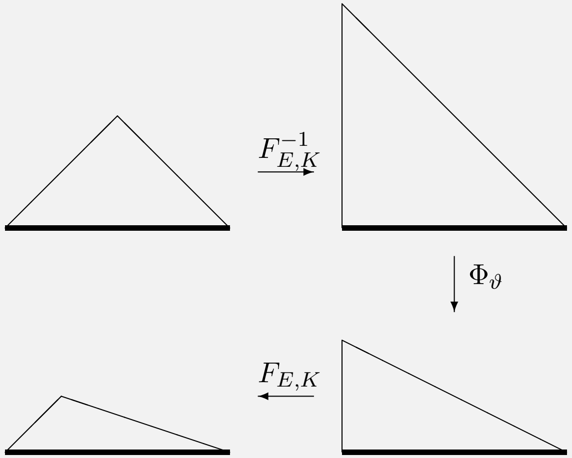
Proof of the Upper Bound

- L^2-representation: $\langle R_h(u_{\mathcal{I}})\,,\,v\rangle = \int_\Omega rv + \int_\Sigma jv$
- Galerkin orthogonality: $S^{1,0}_0(\mathcal{T}) \subset \ker R_h(u_{\mathcal{I}})$
- Quasi-interpolation error estimate: $\|v - I_{\mathcal{T}} v\|_K \le c\alpha_K\, |||v|||_{\widetilde{\omega}_K}$
- Trace inequality: $\|v\|_E^2 \le \frac{|E|}{|K|}\|v\|_K^2 + \frac{2h_K|E|}{|K|}\|v\|_K \|\nabla v\|_K$

Proof of the Lower Bound

- Insert $\psi_K R_K$ in L^2-representation with standard element cut-off functions ψ_K.
- Insert $\psi_{E,\vartheta} R_E$ in L^2-representation with squeezed face cut-off functions $\psi_{E,\vartheta}$ and $\vartheta = d^{\frac{1}{2}} h_E^{-1} \alpha_E = \min\{1, d^{\frac{1}{2}} h_E^{-1} r^{-\frac{1}{2}}\}$.

Squeezed Face Cut-off Function



Estimation of the Temporal Residual

- Recall $R_\tau(u_\mathcal{I}) = \frac{t_n - t}{\tau_n}\rho^n$ with

$$\langle \rho^n, v\rangle = (d\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), \nabla v) + (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v) + (r(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v).$$

- Upper bound:

$$|||\rho^n|||_* \le \{|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}||| + |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*\}$$

- Follows from definition of ρ^n and $|||\cdot|||_*$.
- Lower bound:

$$\frac{1}{3}\{|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}||| + |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*\} \le |||\rho^n|||_*$$

Proof of the Lower Bound

- Set $w^n = u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}$ and choose $v \in H^1_0(\Omega)$ with $|||v||| = |||\mathbf{a}\cdot\nabla w^n|||_*$ and $(\mathbf{a}\cdot\nabla w^n, v) = |||\mathbf{a}\cdot\nabla w^n|||^2_*$
- Insert $\frac{1}{2}w^n + \frac{1}{2}v$ in the definition of ρ^n:

$$\begin{aligned}&\langle \rho^n, \frac{1}{2}w^n + \frac{1}{2}v\rangle \\ &= \underbrace{\frac{1}{2}(d\nabla w^n, \nabla w^n) + \frac{1}{2}(rw^n, w^n)}_{=\frac{1}{2}|||w^n|||^2} + \underbrace{\frac{1}{2}(\mathbf{a}\cdot\nabla w^n, w^n)}_{=0} \\ &\quad + \underbrace{\frac{1}{2}(d\nabla w^n, \nabla v) + \frac{1}{2}(rw^n, v)}_{\ge -\frac{1}{2}|||w^n|||\,|||\mathbf{a}\cdot\nabla w^n|||_*} + \underbrace{\frac{1}{2}(\mathbf{a}\cdot\nabla w^n, v)}_{=\frac{1}{2}|||\mathbf{a}\cdot\nabla w^n|||^2_*}\end{aligned}$$

Estimation of the Convective Derivative I

- Assume that $\|\mathbf{a}\|_\infty \le c_c d$.
- Friedrichs' inequality implies

$$\begin{aligned}(\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), v) &\le \|\mathbf{a}\|_\infty \|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\|\|v\| \\ &\le \|\mathbf{a}\|_\infty \|\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})\| c_\Omega \|\nabla v\|\end{aligned}$$

- Hence $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_* \le c_c c_\Omega |||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}|||$ and $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ is equivalent to $|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}|||$.

Estimation of the Convective Derivative II

- Assume that $\|\mathbf{a}\|_\infty \gg d$.
- Consider the auxiliary problem

$$d(\nabla v^n_{\mathcal{T}_n}, \nabla\varphi) + r(v^n_{\mathcal{T}_n}, \varphi) = (\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}), \varphi) \qquad (*)$$

  with variational and discrete solutions Φ and $\Phi_{\mathcal{T}_n}$.
- Then $|||\Phi||| = |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$, $|||\Phi_{\mathcal{T}_n}||| \le |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ and therefore

$$\begin{aligned}\frac{1}{3}\big\{|||\Phi_{\mathcal{T}_n}||| + |||\Phi - \Phi_{\mathcal{T}_n}|||\big\} &\le |||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_* \\ &\le |||\Phi_{\mathcal{T}_n}||| + |||\Phi - \Phi_{\mathcal{T}_n}|||.\end{aligned}$$

- Transition condition implies that $|||\Phi - \Phi_{\mathcal{T}_n}|||$ is equivalent to every robust, e.g. residual, error indicator $\eta^n_{\mathcal{T}}$ for $(*)$.
- Hence $|||\mathbf{a}\cdot\nabla(u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}})|||_*$ is equivalent to $|||\Phi_{\mathcal{T}_n}||| + \eta^n_{\mathcal{T}}$.

A Posteriori Error Estimate

- Define the space-time error estimator by:

$$\eta^n = \tau_n^{\frac12}\Big[\underbrace{\big(\eta^n_{\mathcal{T}_n}\big)^2}_{\text{spatial}} + \underbrace{|||u^n_{\mathcal{T}_n} - u^{n-1}_{\mathcal{T}_{n-1}}|||^2 + \big(\eta^n_{\mathcal{T}}\big)^2}_{\text{temporal}}\Big]^{\frac12}.$$

- Then

$$\|e\|_{X(0,T)} \le c^*\Big\{\|u_0 - \pi_0 u_0\|^2 + \sum_{n=1}^{N_{\mathcal{I}}}(\eta^n)^2\Big\}^{\frac12},$$

$$\eta^n \le c_*\|e\|_{X(t_{n-1},t_n)}.$$

- $c_* c^*$ only depends on the polynomial degrees and the shape parameters of the partitions $\widetilde{\mathcal{T}}_n$.

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