A posteriori error estimates for linear parabolic equations

R. Verfürth

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany

E-mail address: rv@num1.ruhr-uni-bochum.de

Date: October 2004

Summary: We consider discretizations of linear parabolic equations by A-stable θ-schemes in time and conforming finite elements in space. For these discretizations we derive a residual a posteriori error estimator. The estimator yields upper bounds on the error which are global in space and time and lower bounds that are global in space and local in time. The error estimates are fully robust in the sense that the ratio between upper and lower bounds is uniformly bounded in time, does not depend on any step-size in space or time nor on any relation between these both, and is uniformly bounded with respect to the relative sizes of the coefficients of the differential operator. The results cover a varity of regimes ranging in particular from diffusion dominated ones to convection dominated ones.

Key words: a posteriori error estimates; linear parabolic equations; θ-scheme; space-time finite elements

AMS Subject classification: 65N30, 65N15, 65J15

1. Introduction

We consider linear parabolic equations

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(D\nabla u) + \underline{c} \cdot \nabla u + ru &= f && \text{in } \Omega \times (0,T] \\ u &= 0 && \text{on } \Gamma_D \times (0,T] \\ \underline{n} \cdot D\nabla u &= g && \text{on } \Gamma_N \times (0,T] \\ u &= u_0 && \text{in } \Omega \end{aligned} \tag{1.1}$$

in a bounded space-time cylinder with a polygonal cross-section $\Omega \subset \mathbb{R}^d$, $d \geq 2$, having a Lipschitz boundary Γ consisting of two disjoint parts Γ_D and Γ_N. The final time T is arbitrary, but kept fixed in what follows.

We assume that the data satisfy the following conditions:

(A1) The coefficients $D \in \mathbb{R}^{d\times d}$, $\underline{c} \in \mathbb{R}^d$, and $r \in \mathbb{R}$ are constant.

(A2) The diffusion-coefficient D is symmetric positive definite and isotropic, i.e.

$$\lambda = \min_{z \in \mathbb{R}^d \setminus \{0\}} \frac{z^T D z}{z^T z} > 0$$

and

$$\kappa = \lambda^{-1} \max_{z \in \mathbb{R}^d \setminus \{0\}} \frac{z^T D z}{z^T z} = O(1).$$

(A3) The reaction term r is non-negative.

(A4) The Dirichlet boundary Γ_D has positive $(d-1)$-dimensional measure and includes the inflow boundary $\{x \in \Gamma : \underline{c} \cdot \underline{n}(x) < 0\}$.

Assumption (A1) is made to simplify the exposition. In Section 9 we will deal with variable coefficients. Assumptions (A1) – (A4) guarantee that problem (1.1) is a well-posed parabolic problem. The parameter κ is introduced in order to stress that the ratio of the constants in the error estimates depends on the condition number of the diffusion matrix. If this condition number is exceedingly large, length scales such as element diameters must be measured in a diffusion-dependent metric in order to recover robust estimates (cf. [3] in the context of elliptic equations). Assumptions (A1) – (A4) cover a wide range of different regimes:

dominant diffusion: $|\underline{c}| \le c_c \lambda$ and $r \le c_r \lambda$ with constants c_c, c_r of order 1;

dominant reaction: $|\underline{c}| \le c_c \lambda$ and $r \gg \lambda$ with constant c_c of order 1;

dominant convection: $|\underline{c}| \gg \lambda$.

Thus the present analysis unifies the techniques of [11] and [12] which are devoted to the diffusion-dominated and convection-dominated regimes, respectively.

We use the A-stable θ-schemes for the time-discretization of problem (1.1). The spatial discretization is based on standard conforming finite element spaces using the standard Galerkin formulation or a stabilized SUPG-scheme. For this space-time discretization we analyze a residual error estimator and establish upper and lower bounds for the error. The upper bounds are global with respect to space and time; the lower bounds are global with respect to space and local with respect to time. The ratio of upper and lower bounds is uniformly bounded with respect to any meshsize, to the final time, and – most important – to ratios of the parameters λ, $|\underline{c}|$ and r. Thus the error estimates are fully robust. When dealing with dominant convection, the present estimator requires the solution of an auxiliary discrete stationary reaction-diffusion problem at each time-level. This is the price that we must pay for obtaining bounds that are uniform with respect to $|\underline{c}|/\lambda$. The computational effort for evaluating the error estimator then is comparable to an additional time-step for each time-level.

The error estimator consists of two ingredients: a spatial error indicator and a temporal one. These can be used to monitor the space- and time-adaptivity seperately. The spatial error indicator is a standard residual estimator corresponding to the semi-discretization in time of (1.1). It could be replaced by any other reliable and efficient error estimator for this type of problems such as, e.g., estimators based on the solution of auxiliary local (in space) discrete problems.

The present results should be compared with older ones in [8]:

(1) Here, we consider standard time discretizations which in particular cover the implicit Euler and Crank-Nicolson schemes. In [8], we used a non-standard time discretization which could be interpreted as an implicit Runge-Kutta method and which covered the Crank-Nicolson scheme as method of lowest order.
(2) Here, the ratio of the upper and lower error bounds is independent of any mesh-size in space or time and of any relation between these parameters. In [8], this ration is proportional to $1 + h^2\tau^{-1} + h^{-2}\tau$ where h and τ denote the local mesh-sizes in space and time respectively.
(3) In [8], we considered genaral non-linear parabolic problems. Here we confine ourselves to the linear case. An extension to non-linear parabolic equations is under way [13].

The article is organized as follows. In Section 2 we introduce some function spaces and norms. Section 3 is devoted to the finite element discretization. Using energy estimates we prove in Section 4 that the error is equivalent to a residual which is defined in a suitable dual space. This residual is split into three parts: one corresponding to the approximation of the data, a contribution corresponding to a spatial error, and a part corresponding to a temporal error. The latter can be further decomposed into a diffusive and a convective part. In Section 5 we derive upper and lower bounds for the spatial part of the residual. The temporal part is treated in Section 6. Combining these results we obtain in Section 7 a first error estimator. This estimator yields upper and lower bounds on the error and is fully robust in the sense described above. However, it is not suited for practical computations since it incorporates a dual norm of the convective derivative of the finite element solution. This contribution is due to the convective part of the temporal residual. The results of Section 7 show that sharp upper and lower bounds with parameter independent constants for this term are mandatory for obtaining a robust and computable a posteriori error estimator. This task is achieved in Section 8. As long as the convection is not dominant we can simply bound the dual norm by a standard L^2-norm that is computable. This leads to Theorem 8.1 that covers the cases of dominant diffusion and of dominant reaction. In the case of dominant convection, we bound the critical dual norm by computable quantites based on the solution of a discrete stationary reaction-diffusion problem at each time-level. This leads to Theorem 8.3. In Section 9 we finally present the modifications that are necessary to treat variable coefficients.

2. Function spaces

For any bounded open subset ω of Ω with Lipschitz boundary γ, we denote by $H^k(\omega)$, $k \in \mathbb{N}$, $L^2(\omega) = H^0(\omega)$, and $L^2(\gamma)$ the usual Sobolev and Lebesgue spaces equipped with the standard norms $\|.\|_{k;\omega} = \|.\|_{H^k(\omega)}$ and $\|.\|_{0;\gamma} = \|.\|_{L^2(\gamma)}$ (cf. [1]). Similarly,

$(.,.)_\omega$ and $(.,.)_\gamma$ denote the scalar products of $L^2(\omega)$ and $L^2(\gamma)$, respectively. If $\omega = \Omega$, we will omit the index Ω.

Set

$$H^1_D(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}. \tag{2.1}$$

We equip $H^1_D(\Omega)$ with the norm

$$|||v||| = \left\{\lambda \|\nabla v\|_0^2 + r\|v\|_0^2\right\}^{1/2}, \tag{2.2}$$

where λ is the smallest eigenvalue of the diffusion matrix D defined in Assumption (A2) of §1. Due to Assumptions (A2) – (A4) this is the natural energy norm of problem (1.1). The dual space of $H^1_D(\Omega)$ is denoted by $H^{-1}(\Omega)$ and is equipped with the norm

$$|||\varphi|||_* = \sup_{v \in H^1_D(\Omega)\setminus\{0\}} \frac{\langle \varphi, v\rangle}{|||v|||}, \tag{2.3}$$

where $\langle .,.\rangle$ denotes the corresponding duality pairing.

$H^{1/2}(\Gamma_N)$ denotes the space of Γ_N-traces of H^1-functions and is equipped with the trace norm induced by the energy norm, i.e.

$$\|\varphi\|_{H^{1/2}(\Gamma_N)} = \inf\left\{|||v||| : v \in H^1_D(\Omega)\ ,\ v = \varphi \text{ on } \Gamma_N\right\}.$$

$H^{-1/2}(\Gamma_N)$ denotes the dual space of $H^{1/2}(\Gamma_N)$ and is equipped with the corresponding dual norm. Thus the norms of $H^{1/2}(\Gamma_N)$ and $H^{-1/2}(\Gamma_N)$ depend on the energy norm and consequently on the parameters λ and r.

For any seperable Banach space V and any two numbers $a < b$ we denote by $L^2(a,b;V)$ and $L^\infty(a,b;V)$ the spaces of measurable functions u defined on (a,b) with values in V such that the function $t \to \|u(.,t)\|_V$ is square integrable respectively essentially bounded. These are Banach spaces equipped with the norms

$$\begin{aligned} \|u\|_{L^2(a,b;V)} &= \left\{\int_a^b \|u(.,t)\|_V^2 dt\right\}^{1/2}, \\ \|u\|_{L^\infty(a,b;V)} &= \operatorname*{ess.sup}_{a<t<b} \|u(.,t)\|_V \end{aligned}$$

(cf. [4, Vol. 5, Chap. XVIII, §1]). For abbreviation we introduce the space

$$\begin{aligned} X(a,b) = \big\{u \in L^2(a,b;H^1_D(\Omega)) \cap L^\infty(a,b;L^2(\Omega)) :& \\ \partial_t u + \underline{c}\cdot\nabla u \in L^2(a,b;H^{-1}(\Omega))\big\}& \end{aligned} \tag{2.4}$$

and equip it with its graph norm

$$\begin{aligned} \|u\|_{X(a,b)} = \Big\{&\operatorname*{ess.sup}_{a<t<b} \|u(.,t)\|_0^2 + \int_a^b |||u(.,t)|||^2 dt \\ &+ \int_a^b |||(\partial_t u + \underline{c}\cdot\nabla u)(.,t)|||_*^2 dt\Big\}^{1/2}. \end{aligned} \tag{2.5}$$

Here the derivative $\partial_t u$ has to be understood in the distributional sense [4, loc. cit.].

The weak form of problem (1.1) consists in finding $u \in L^2(0,T;H^1_D(\Omega))$ such that $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$, $u(.,0) = u_0$ in $H^{-1}(\Omega)$, and for almost every $t \in (0,T)$ and all $v \in H^1_D(\Omega)$

$$(\partial_t u, v) + (D\nabla u, \nabla v) + (\underline{c} \cdot \nabla u, v) + (ru, v) = (f, v) + (g, v)_{\Gamma_N}. \qquad (2.6)$$

Assumptions (A1) – (A4) imply that problem (2.6) admits a unique solution (cf. e.g. [2], [4]).

For later use we note that integration by parts and Assumptions (A1) – (A4) imply

$$(D\nabla v, \nabla v) + (\underline{c} \cdot \nabla v, v) + (rv, v) \geq |||v|||^2 \quad \forall v \in H^1_D(\Omega). \qquad (2.7)$$

Similarly, Assumptions (A1) and (A2) and definition (2.2) yield

$$(D\nabla v, \nabla w) + (rv, w) \leq \kappa |||v|||\ |||w||| \quad \forall v, w \in H^1_D(\Omega). \qquad (2.8)$$

3. Finite element discretization

For the discretization we choose an integer $N \geq 1$ and intermediate times $0 = t_0 < t_1 < \ldots < t_N = T$ and set $\tau_n = t_n - t_{n-1}$, $1 \leq n \leq N$. With each intermediate time t_n, $0 \leq n \leq N$, we associate a partition $\mathcal{T}_{h,n}$ of Ω and a corresponding finite element space $X_{h,n}$. These have to satisfy the following conditions:

(1) *Affine equivalence:* every element $K \in \mathcal{T}_{h,n}$ can be mapped by an invertible affine mapping onto the standard reference d-simplex or the standard unit cube in $\mathbb{R}^d$.

(2) *Admissibility:* any two elements are either disjoint or share a vertex, or a complete edge, or (if $d = 3$) a complete face.

(3) *Shape-regularity:* for any element K the ratio of its diameter h_K to the diameter ρ_K of the largest inscribed ball is bounded uniformly with respect to all partitions $\mathcal{T}_{h,n}$ and to N.

(4) *Transition condition:* for $1 \leq n \leq N$ there is an affinely equivalent, admissible, and shape-regular partition $\widetilde{\mathcal{T}}_{h,n}$ such that it is a refinement of both $\mathcal{T}_{h,n}$ and $\mathcal{T}_{h,n-1}$ and such that

$$\sup_{1 \leq n \leq N} \sup_{K \in \widetilde{\mathcal{T}}_{h,n}} \sup_{K' \in \mathcal{T}_{h,n}; K \subset K'} \frac{h_{K'}}{h_K} < \infty.$$

(5) Each $X_{h,n}$ is a subset of $H^1_D(\Omega)$ and consists of continuous functions which are piecewise polynomials, the degrees being bounded uniformly with respect to all partitions $\mathcal{T}_{h,n}$ and to N.

(6) Each $X_{h,n}$ contains the space of continuous, piecewise linear finite elements corresponding to $\mathcal{T}_{h,n}$.

Condition (1) restricts quadrilateral elements to parallelograms and cubic elements to parallelepipeds. In two dimensions, triangular and quadrilateral elements may be mixed. In three dimensions this is also possible if one adds prismatic elements. Condition (2) excludes hanging nodes. Condition (3) is a standard one and allows for highly refined meshes. However, it excludes anisotropic elements. Condition (4) is due to the simultaneous presence of finite element functions defined on different grids. In practice the partition $\mathcal{T}_{h,n}$ is usually obtained from $\mathcal{T}_{h,n-1}$ by a combination of refinement and of coarsening. In this case Condition (4) only restricts the coarsening. It must not be too abrupt nor too strong. Condition (4) is not needed when using the implicit Euler scheme, i.e. $\theta = 1$.

We choose a parameter $\theta \in [\frac{1}{2}, 1]$ and keep it fixed in what follows. For every time-level $n \geq 1$ we introduce the abbreviation

$$f^{n\theta} = \theta f(., t_n) + (1-\theta) f(., t_{n-1}) \;,\; g^{n\theta} = \theta g(., t_n) + (1-\theta) g(., t_{n-1}).$$

Then the space-time discretization of problem (1.1) consists in finding $u_h^n \in X_{h,n}$, $0 \leq n \leq N$, such that

$$u_h^0 = \pi_0 u_0 \tag{3.1}$$

and, for $n = 1, \ldots, N$, and all $v_h \in X_{h,n}$

$$\begin{aligned}
\Big(\frac{u_h^n - u_h^{n-1}}{\tau_n}, v_h\Big) &+ \big(D\nabla(\theta u_h^n + (1-\theta)u_h^{n-1}), \nabla v_h\big) \\
&+ \big(\underline{c} \cdot \nabla(\theta u_h^n + (1-\theta)u_h^{n-1}), v_h\big) \\
&+ \big(r(\theta u_h^n + (1-\theta)u_h^{n-1}), v_h\big) \\
&+ \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \delta_K \Big(\frac{u_h^n - u_h^{n-1}}{\tau_n} - \operatorname{div}(D\nabla(\theta u_h^n + (1-\theta)u_h^{n-1})) \\
&\qquad + \underline{c} \cdot \nabla(\theta u_h^n + (1-\theta)u_h^{n-1}) \\
&\qquad + r(\theta u_h^n + (1-\theta)u_h^{n-1}) \;,\; \underline{c} \cdot \nabla v_h\Big)_K \\
&= (f^{n\theta}, v_h) + (g^{n\theta}, v_h)_{\Gamma_N} + \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \delta_K (f^{n\theta}, \underline{c} \cdot \nabla v_h)_K .
\end{aligned} \tag{3.2}$$

Here, π_0 denotes the L^2-projection onto $X_{h,0}$. The δ_K are non-negative stabilization parameters. The choice $\delta_K = 0$ for all K yields the standard Galerkin discretization; the choice $\delta_K > 0$ for all K corresponds to the SUPG-discretizations (cf., e.g., [5], [6]). In what follows we will always assume that

$$\delta_K |\underline{c}| \leq h_K \quad \forall K \in \widetilde{\mathcal{T}}_{h,n},\; 0 \leq n \leq N. \tag{3.3}$$

This condition is satisfied for all choices of δ_K used in practice.

Assumptions (A2) – (A4), and (3.3) and standard arguments for SUPG-discretizations (cf., e.g., [5], [6]) imply that problems (3.1), (3.2) admit a unique solution $(u_h^n)_{0\le n\le N}$. With this sequence we associate the function $u_{h,\tau}$ which is *piecewise affine* on the time-intervals $[t_{n-1}, t_n]$, $1 \le n \le N$, and which equals u_h^n at time t_n, $0 \le n \le N$.

For abbreviation we denote by $f_{h,\tau}$ and $g_{h,\tau}$ functions that are *piecewise constant* on the time-intervals $(t_{n-1}, t_n]$, $1 \le n \le N$, and which equal the L^2-projection of $f^{n\theta}$ onto the space $X_{h,n}$ and the L^2-projection of $g^{n\theta}$ onto the space of traces of $X_{h,n}$-functions respectively. With this notation we can replace in (3.2) the functions $f^{n\theta}$ and $g^{n\theta}$ by $f_{h,\tau}$ and $g_{h,\tau}$, respectively.

4. The equivalence of error and residual

With the function $u_{h,\tau}$ defined by the solution of problems (3.1), (3.2) we associate the residual $R(u_{h,\tau}) \in L^2(0,T;H^{-1}(\Omega))$ via

$$\begin{aligned}\langle R(u_{h,\tau}), v\rangle =&(f,v) + (g,v)_{\Gamma_N} - (\partial_t u_{h,\tau}, v) - (D\nabla u_{h,\tau}, \nabla v)\\ &- (\underline{c}\cdot\nabla u_{h,\tau}, v) - (r u_{h,\tau}, v)\end{aligned} \tag{4.1}$$

for all $v \in H^1_D(\Omega)$. The following lemma shows that this residual and the error $u - u_{h,\tau}$ are equivalent. Its proof is based on standard energy estimates. Recall that $H^1_D(\Omega)$ and $H^{-1}(\Omega)$ are equipped with the energy-norm $|||.|||$ and the dual norm $|||.|||_*$ respectively.

4.1 Lemma. *For all $w \in L^2(0,T;H^1_D(\Omega))$ the following lower bound on the error holds*

$$\int_0^T \langle R(u_{h,\tau}), w\rangle dt \le \sqrt{1+\kappa^2}\|u-u_{h,\tau}\|_{X(0,T)}\|w\|_{L^2(0,T;H^1_D(\Omega))}, \tag{4.2}$$

where κ is the constant of Assumption (A2). Conversely, for all n between 1 and N, the error can be bounded from above by

$$\begin{aligned}\|u-u_{h,\tau}\|_{X(0,t_n)} \le \Big\{&2(1+\kappa^2)\|u_0-\pi_0 u_0\|_0^2\\ &+2(2+\kappa^2)\|R(u_{h,\tau})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}\Big\}^{1/2}.\end{aligned} \tag{4.3}$$

Proof. Equations (2.6) and (3.2) imply for all $v \in H^1_D(\Omega)$ that

$$\begin{aligned}(\partial_t(u-u_{h,\tau}), v)+&(D\nabla(u-u_{h,\tau}), \nabla v)\\ &+(\underline{c}\cdot\nabla(u-u_{h,\tau}), v) + (r(u-u_{h,\tau}), v) = \langle R(u_{h,\tau}), v\rangle.\end{aligned} \tag{4.4}$$

This identity, definitions (2.2) and (2.3) of the norms $|||.|||$ and $|||.|||_*$, and inequality (2.8) yield for all $0 < t < T$ and all $v \in H^1_D(\Omega)$ the estimate

$$\begin{aligned}\langle R(u_{h,\tau}), v\rangle \le& |||(\partial_t(u-u_{h,\tau}) + \underline{c}\cdot\nabla(u-u_{h,\tau}))(.,t)|||_* |||v||| \\ &+ \kappa|||(u-u_{h,\tau})(.,t)|||\ |||v|||.\end{aligned}$$

Taking into account the definitions (2.4), (2.5) of $X(0;T)$ and of its norm, this estimate proves the bound (4.2).

To prove estimate (4.3) we choose an integer n between 1 and N and a time t between 0 and t_n and insert $v = (u-u_{h,\tau})(.,t)$ in equation (4.4). Taking into account inequality (2.7), this gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(u-u_{h,\tau})(.,t)\|_0^2 + |||(u-u_{h,\tau})(.,t)|||^2 \\ &\quad\le(\partial_t(u-u_{h,\tau})(.,t),(u-u_{h,\tau})(.,t)) + (D\nabla(u-u_{h,\tau})(.,t),\nabla(u-u_{h,\tau})(.,t)) \\ &\qquad + (\underline{c}\cdot\nabla(u-u_{h,\tau})(.,t),(u-u_{h,\tau})(.,t)) + (r(u-u_{h,\tau})(.,t),(u-u_{h,\tau})(.,t)) \\ &\quad=\langle R(u_{h,\tau})(.,t),(u-u_{h,\tau})(.,t)\rangle \\ &\quad\le|||R(u_{h,\tau})(.,t)|||_*|||(u-u_{h,\tau})(.,t)||| \\ &\quad\le\frac{1}{2}|||R(u_{h,\tau})(.,t)|||_*^2 + \frac{1}{2}|||(u-u_{h,\tau})(.,t)|||^2\end{aligned}$$

and thus

$$\frac{d}{dt}\|(u-u_{h,\tau})(.,t)\|_0^2 + |||(u-u_{h,\tau})(.,t)|||^2 \le |||R(u_{h,\tau})(.,t)|||_*^2.$$

Integrating this estimate from 0 to t implies

$$\begin{aligned}\|(u-u_{h,\tau})(.,t)\|_0^2 - \|u_0-\pi_0u_0\|_0^2 + \int_0^t |||(u-u_{h,\tau})(.,s)|||^2\, ds \\ \le\|R(u_{h,\tau})\|^2_{L^2(0,t;H^{-1}(\Omega))} \\ \le\|R(u_{h,\tau})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}.\end{aligned}$$

Since $t \in (0,t_n]$ was arbitrary, this yields

$$\|u-u_{h,\tau}\|^2_{L^\infty(0,t_n;L^2(\Omega))} \le \|u_0-\pi_0u_0\|_0^2 + \|R(u_{h,\tau})\|^2_{L^2(0,t_n;H^{-1}(\Omega))} \tag{4.5}$$

and

$$\|u-u_{h,\tau}\|^2_{L^2(0,t_n;H^1_D(\Omega))} \le \|u_0-\pi_0u_0\|_0^2 + \|R(u_{h,\tau})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}. \tag{4.6}$$

Equation (4.4) and estimate (2.8), on the other hand, imply

$$|||\partial_t(u-u_{h,\tau}) + \underline{c}\cdot\nabla(u-u_{h,\tau})|||_* \le |||R(u_{h,\tau})|||_* + \kappa|||u-u_{h,\tau}|||.$$

Taking the square of this inequality, integrating from 0 to t_n, and inserting estimate (4.6) we arrive at

$$\begin{aligned}
&\|\partial_t(u-u_{h,\tau})+\underline{c}\cdot\nabla(u-u_{h,\tau})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}\\
&\qquad\le 2\|R(u_{h,\tau})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}+2\kappa^2\|u-u_{h,\tau}\|^2_{L^2(0,t_n;H^1_D(\Omega))}\\
&\qquad\le 2\kappa^2\|u_0-\pi_0u_0\|_0^2\\
&\qquad\quad+2(1+\kappa^2)\|R(u_{h,\tau})\|^2_{L^2(0,t_n;H^{-1}(\Omega))}.
\end{aligned}\tag{4.7}$$

Combining estimates (4.5) – (4.7) proves the bound (4.3). □

The subsequent analysis relies on an appropriate decomposition of the residual $R(u_{h,\tau})$. To this end we recall the definition of the functions $f_{h,\tau}$ and $g_{h,\tau}$ at the end of §3 and define a temporal residual $R_\tau(u_{h,\tau})\in L^2(0,T;H^{-1}(\Omega))$ and a spatial residual $R_h(u_{h,\tau})\in L^2(0,T;H^{-1}(\Omega))$ by setting – for all $v\in H^1_D(\Omega)$ and all $1\le n\le N$ –

$$\begin{aligned}
\langle R_\tau(u_{h,\tau}),v\rangle =&(D\nabla(\theta u_h^n+(1-\theta)u_h^{n-1}-u_{h,\tau}),\nabla v)\\
&+(\underline{c}\cdot\nabla(\theta u_h^n+(1-\theta)u_h^{n-1}-u_{h,\tau}),v)\\
&+(r(\theta u_h^n+(1-\theta)u_h^{n-1}-u_{h,\tau}),v)\qquad\text{on } (t_{n-1},t_n]
\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}
\langle R_h(u_{h,\tau}),v\rangle =&(f_{h,\tau},v)+(g_{h,\tau},v)_{\Gamma_N}-(\frac{u_h^n-u_h^{n-1}}{\tau_n},v)\\
&-(D\nabla(\theta u_h^n+(1-\theta)u_h^{n-1}),\nabla v)\\
&-(\underline{c}\cdot\nabla(\theta u_h^n+(1-\theta)u_h^{n-1}),v)\\
&-(r(\theta u_h^n+(1-\theta)u_h^{n-1}),v)\qquad\text{on } (t_{n-1},t_n].
\end{aligned}\tag{4.9}$$

The discretization of the data is taken into account by a data-residual $R_D(u_{h,\tau})\in L^2(0,T;H^{-1}(\Omega))$ which is defined by

$$\langle R_D(u_{h,\tau}),v\rangle=(f-f_{h,\tau},v)+(g-g_{h,\tau},v)_{\Gamma_N}.\tag{4.10}$$

Since $\partial_t u_{h,\tau}$ is piecewise constant and equals $\frac{u_h^n-u_h^{n-1}}{\tau_n}$ on $(t_{n-1},t_n]$, we obtain the decomposition

$$R(u_{h,\tau})=R_D(u_{h,\tau})+R_\tau(u_{h,\tau})+R_h(u_{h,\tau}).\tag{4.11}$$

5. Estimation of the spatial residual

For the estimation of the spatial residual $R_h(u_{h,\tau})$ we need some additional notations. We denote by $\widetilde{\mathcal{E}}_{h,n}$, $1\le n\le N$, the set of all edges (if $d=2$) respectively faces (if $d=3$) of $\widetilde{\mathcal{T}}_{h,n}$. With each edge or face $E\in\widetilde{\mathcal{E}}_{h,n}$ we associate a unit vector $\underline{n}_E$

orthogonal to E such that it points to the outward of Ω if E lies on the boundary. For every edge or face E that is not contained in the boundary Γ we denote by $[.]_E$ the jump across E in direction $\underline{n}_E$. The quantity $[.]_E$ of course depends on the orientation of $\underline{n}_E$, but quantities of the form $[\underline{n}_E \cdot .]_E$ are independent thereof. With each edge respectively face we associate the set ω_E which is the union of the elements that share E.

We recall the definition of the functions $f_{h,\tau}$ and $g_{h,\tau}$ at the end of §3 and define element residuals R_K, $K \in \widetilde{\mathcal{T}}_{h,n}$, $1 \le n \le N$, by

$$\begin{aligned} R_K =& f_{h,\tau} - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \operatorname{div}(D\nabla(\theta u_h^n + (1-\theta)u_h^{n-1})) \\ & - \underline{c} \cdot \nabla(\theta u_h^n + (1-\theta)u_h^{n-1}) - r(\theta u_h^n + (1-\theta)u_h^{n-1}), \end{aligned} \tag{5.1}$$

and edge respectively face residuals R_E, $E \in \widetilde{\mathcal{E}}_{h,n}$, $1 \le n \le N$, by

$$R_E = \begin{cases} -\left[\underline{n}_E \cdot D\nabla(\theta u_h^n + (1-\theta)u_h^{n-1})\right]_E & \text{if } E \not\subset \Gamma, \\ g_{h,\tau} - \underline{n}_E \cdot D\nabla(\theta u_h^n + (1-\theta)u_h^{n-1}) & \text{if } E \subset \Gamma_N, \\ 0 & \text{if } E \subset \Gamma_D. \end{cases} \tag{5.2}$$

Here, of course, $(u_h^n)_{0 \le n \le N}$ denotes the solution of problems (3.1) and (3.2).

For every n between 1 and N we denote by $\mathcal{N}_{h,n}$ the set of all element vertices in $\mathcal{T}_{h,n}$ that do not lie on the Dirichlet boundary Γ_D. With every vertex $x \in \mathcal{N}_{h,n}$ we associate the nodal bases function λ_x which is uniquely defined by the properties

$$\lambda_{x|K} \in R_1(K) \quad \forall K \in \mathcal{T}_{h,n}, \quad \lambda_x(y) = 0 \quad \forall y \in \mathcal{N}_{h,n} \backslash \{x\}, \quad \lambda_x(x) = 1.$$

Here, as usual, $R_k(K)$ denotes the set of all polynomials of total degree k, if K is a simplex, and of maximal degree k, if K is a parallelepiped. The support of a nodal bases function λ_x is denoted by ω_x and consists of all elements in $\mathcal{T}_{h,n}$ that share the vertex x. With this notation we can define a Clément-type interpolation operator $I_{h,n} : L^1(\Omega) \longrightarrow \{\varphi \in C(\Omega) : \varphi_{|K} \in R_1(K) \text{ for all } K \in \mathcal{T}_{h,n}, \varphi = 0 \text{ on } \Gamma_D\}$ by (cf. [10])

$$I_{h,n} v = \sum_{x \in \mathcal{N}_{h,n}} \left\{ \frac{1}{|\omega_x|} \int_{\omega_x} v \right\} \lambda_x. \tag{5.3}$$

Here $|\omega_x|$ denotes the d-dimensional Lebesgue-measure of ω_x. Due to Condition (6) of Section 3 the image of $I_{h,n}$ is contained in $X_{h,n}$.

5.1 Lemma. *For every $S \in \widetilde{\mathcal{T}}_{h,n} \cup \widetilde{\mathcal{E}}_{h,n}$, $1 \le n \le N$, denote by h_S its diameter and set*

$$\alpha_S = \min\{h_S \lambda^{-1/2}, r^{-1/2}\}. \tag{5.4}$$

Then the following estimates hold for all n *between* 1 *and* N*, all elements* $K \in \widetilde{\mathcal{T}}_{h,n}$*, all edges respectively faces* E *of* K*, and all function* $v \in H^1_D(\Omega)$

$$\begin{aligned} \|v - I_h v\|_{0;K} &\le c_1 \alpha_K |||v|||_{\tilde{\omega}_K}, \\ \|v - I_h v\|_{0;E} &\le c_2 \lambda^{-1/4} \alpha_E^{1/2} |||v|||_{\tilde{\omega}_K}, \\ |||I_h v|||_K &\le c_3 |||v|||_{\tilde{\omega}_K}. \end{aligned}$$

Here, $\tilde{\omega}_K$ *is the union of all elements in* $\mathcal{T}_{h,n}$ *that share at least a vertex with the element* $K' \in \mathcal{T}_{h,n}$ *that contains* K *and* $|||.|||_A$ *denotes the restriction of* $|||.|||$ *to the measurable set* A*.*

Proof. Taking into account Condition (4) of §3, the proof of Lemma 5.1 follows from Lemma 3.1 in [9] and Proposition 2.1 in [10] with the arguments used in the proof of Lemma 3.2 in [9]. ☐

5.2 Remark. In the case $r = 0$ the minimum in (5.4) of course yields $\alpha_S = \lambda^{-1/2} h_S$ for all S.

For every element $K \in \widetilde{\mathcal{T}}_{h,n}$, $1 \le n \le N$, we denote by $\mathcal{N}_K$ the set of its vertices and set

$$\psi_K = \gamma_K \prod_{x \in \mathcal{N}_K} \lambda_x, \tag{5.5}$$

where the constant γ_K is choosen such that ψ_K equals 1 at the barycentre of K. Note that the support of ψ_K is contained in K and that $\|\psi_K\|_{L^\infty(K)} = 1$.

For every edge respectively face $E \in \widetilde{\mathcal{E}}_{h,n}$, $1 \le n \le N$, we set

$$\theta_E = \min\{\lambda^{1/2} r^{-1/2} h_E^{-1}, 1\} = h_E^{-1} \lambda^{1/2} \alpha_E \tag{5.6}$$

and denote by $\mathcal{N}_E$ the set of its vertices. (Note that $\theta_E = 1$ in the case $r = 0$.) Consider first an edge respectively face E that is not contained in the boundary. It is shared by exactly two elements $K_{E,1}$ and $K_{E,2}$. For $i = 1, 2$ we define an affine transformation $F_i : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ as follows: We first map $K_{E,i}$ onto the reference element such that the image of E is contained in the hyperplane $\{x_d = 0\}$; then we apply the transformation $(x_1, \dots, x_{d-1}, x_d) \longrightarrow (x_1, \dots, x_{d-1}, \theta_E x_d)$; and finally we transform back with the inverse of the affine transformation of the first step. With this definition we set

$$\psi_E = \gamma_E \prod_{x \in \mathcal{N}_E} \lambda_x \circ F_i^{-1} \quad \text{on } K_{E,i}\ , i = 1, 2, \tag{5.7}$$

where the constant γ_E is choosen such that ψ_E equals 1 at the barycentre of E. Note that the support of ψ_E is contained in $F_1(K_{E,1}) \cup F_2(K_{E,2}) \subset K_{E,1} \cup K_{E,2} = \omega_E$ and that $\|\psi_E\|_{L^\infty(E)} = 1$.

If an edge respectively face E is contained in the Neumann boundary Γ_N the defintion of ψ_E is modified in the obvious way taking into account that now E is the face of exactly one element K_E.

5.3 Lemma. *The following estimates hold for all n between 1 and N, all elements $K \in \widetilde{\mathcal{T}}_{h,n}$, all polynomials $v \in R_k(K)$, all edges respectively faces $E \in \widetilde{\mathcal{E}}_{h,n}$, and all polynomials $\sigma \in R_k(E)$*

$$
\begin{aligned}
(v, \psi_K v)_K &\geq c_4 \|v\|_{0;K}^2, \\
|||\psi_K v|||_K &\leq c_5 \alpha_K^{-1} \|v\|_{0;K}, \\
(\sigma, \psi_E \sigma)_E &\geq c_6 \|\sigma\|_{0;E}^2, \\
|||\psi_E \sigma|||_{\omega_E} &\leq c_7 \lambda^{1/4} \alpha_E^{-1/2} \|\sigma\|_{0;E}, \\
\|\psi_E \sigma\|_{0;\omega_E} &\leq c_8 \lambda^{1/4} \alpha_E^{1/2} \|\sigma\|_{0;E}.
\end{aligned}
$$

Here, a polynomial σ defined on an edge respectively face E is continued in the canonical way to a polynomial defined on $\mathbb{R}^d$. The constants $c_4, \ldots, c_8$ only depend on the polynomial degree k and on the ratio h_K/ρ_K.
Proof. The estimates are proven with the same arguments as in the proof of Lemma 3.3 in [9]. For parallelepipeds one only has to take into account that the transformation to the unit cube is affine and thus has a constant Jacobian. □

With these preparations we are now ready to bound the spatial residual.

5.4 Lemma. *For every n between 1 and N define a spatial error indicator η_h^n by*

$$
\eta_h^n = \left\{ \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \alpha_K^2 \|R_K\|_{L^2(K)}^2 + \sum_{E \in \widetilde{\mathcal{E}}_{h,n}} \lambda^{-1/2} \alpha_E \|R_E\|_{L^2(E)}^2 \right\}^{1/2}. \tag{5.8}
$$

Then, on each interval $(t_{n-1}, t_n]$, the spatial residual is bounded from above by

$$
|||R_h(u_{h,\tau})|||_* \leq c^\dagger \eta_h^n. \tag{5.9}
$$

Moreover, there are functions $w_n \in H_D^1(\Omega)$ such that on each interval $(t_{n-1}, t_n]$ the spatial residual is bounded from below by

$$
\begin{aligned}
(\eta_h^n)^2 &\leq \langle R_h(u_{h,\tau}), w_n \rangle, \\
|||w_n||| &\leq c_\dagger \eta_h^n.
\end{aligned} \tag{5.10}
$$

The constants $c^\dagger$ and $c_\dagger$ depend on the ratios h_K/ρ_K in Condition (3) of §3. The constant $c^\dagger$ in addition depends on the ratios $h_{K'}/h_K$ in Condition (4) of §3. The constant $c_\dagger$ in addition depends on the maximum of the polynomial degrees of the finite element functions.
Proof. Choose an integer n between 1 and N and keep it fixed in what follows.
Since $\widetilde{\mathcal{T}}_{h,n}$ is a common refinement of $\mathcal{T}_{h,n}$ and $\mathcal{T}_{h,n-1}$, the functions u_h^n and u_h^{n-1} are piecewise polynomials on the elements of $\widetilde{\mathcal{T}}_{h,n}$. Therefore we may integrate by parts

on the elements in $\widetilde{\mathcal{T}}_{h,n}$ and obtain the following L^2-representation of the spatial residual

$$\langle R_h(u_{h,\tau}), v\rangle = \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} (R_K, v)_K + \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} (R_E, v)_E. \tag{5.11}$$

Lemma 5.1 and the Cauchy-Schwarz inequality therefore imply for all $v \in H^1_D(\Omega)$

$$\begin{aligned}\langle R_h(u_{h,\tau}),&v - I_{h,n}v\rangle \\ \le& c|||v|||\Big\{\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \alpha_K^2\|R_K\|^2_{0;K} + \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} \lambda^{-1/2}\alpha_E\|R_E\|^2_{0;E}\Big\}^{1/2}.\end{aligned} \tag{5.12}$$

The constant c only depends on the constants c_1 and c_2 of Lemma 5.1 and on the ratios h_K/ρ_K.
From the definition of problem (3.2) and the definiton (4.9) of the spatial residual we conclude that

$$\langle R_h(u_{h,\tau}), I_{h,n}v\rangle = \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \delta_K(R_K, \underline{c}\cdot\nabla I_{h.n}v)_K.$$

Invoking a standard inverse estimate we obtain for all elements K that

$$\begin{aligned}\|\underline{c}\cdot\nabla I_{h,n}v\|_{0;K} \le& |\underline{c}|\min\{\|\nabla I_{h,n}v\|_{0;K}, c_I h_K^{-1}\|I_{h,n}v\|_{0;K}\} \\ \le& c_I|\underline{c}|h_K^{-1}\alpha_K|||I_{h,n}v|||_K,\end{aligned}$$

where the constant c_I depends on h_K/ρ_K. Lemma 5.1, condition (3.3) and the Cauchy-Schwarz inequality therefore imply

$$\langle R_h(u_{h,\tau}), I_{h,n}v\rangle \le c|||v|||\Big\{\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \alpha_K^2\|R_K\|^2_{0;K}\Big\}^{1/2}. \tag{5.13}$$

Equation (5.11) and estimates (5.12) and (5.13) prove the upper bound (5.9).
For the proof of the lower bound (5.10) we proceed as in the proof of [11, Lemma 5.1] and define the function w_n by

$$w_n = \gamma_1\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \alpha_K^2\psi_K R_K + \gamma_2\sum_{E\in\widetilde{\mathcal{E}}_{h,n}} \lambda^{-1/2}\alpha_E\psi_E R_E. \tag{5.14}$$

The constants γ_1 and γ_2 are arbitrary at present and will be determined below. The subsequent arguments are based on the following observations:

- the supports of the ψ_K are mutually disjoint,
- the support of a ψ_K intersects the support of at most $2d$ different ψ_E's,
- the support of a ψ_E intersects the support of at most two ψ_K's,
- the support of a ψ_E intersects the support of at most $2d-2$ other ψ_E's.

Lemma 5.3 therefore yields

$$
\begin{aligned}
|||w_h|||^2 \leq & \gamma_1^2 \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \alpha_K^4 |||\psi_K R_K|||_K^2 \\
& + 2\gamma_1\gamma_2 \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \left\{ \sum_{E;\omega_E\cap K\neq\emptyset} \alpha_K^2 \lambda^{-1/2}\alpha_E |||\psi_K R_K|||_K |||\psi_E R_E|||_K \right\} \\
& + \gamma_2^2 \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} \left\{ \sum_{E';\omega_E\cap\omega_{E'}\neq\emptyset} \lambda^{-1}\alpha_E\alpha_{E'} |||\psi_E R_E|||_{\omega_E} |||\psi_{E'} R_{E'}|||_{\omega_{E'}} \right\} \\
\leq & (2d+1)\max\{\gamma_1^2,\gamma_2^2\}\max\{c_5,c_7\}\left(\eta_h^n\right)^2 .
\end{aligned}
\tag{5.15}
$$

Since $h_E \leq h_K$ for all edges respectively faces E of any element K, Lemma 5.3 also implies that

$$
\begin{aligned}
\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} & (R_K, w_n)_K + \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} (R_E, w_n)_E \\
= & \gamma_1 \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \alpha_K^2 (R_K, \psi_K R_K)_K + \gamma_2 \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} \lambda^{-1/2}\alpha_E (R_E, \psi_E R_E)_E \\
& + \gamma_2 \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} \left\{ \sum_{K;K\cap\omega_E\neq\emptyset} \lambda^{-1/2}\alpha_E (R_K, \psi_E R_E)_K \right\} \\
\geq & \gamma_1 \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} c_4\alpha_K^2 \|R_K\|_{0;K}^2 + \gamma_2 \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} c_6\lambda^{-1/2}\alpha_E \|R_E\|_{0;E}^2 \\
& - \gamma_2 \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} \left\{ \sum_{K;K\cap\omega_E\neq\emptyset} c_8\lambda^{-1/4}\alpha_E^{1/2}\alpha_K \|R_K\|_{0;K} \|\psi_E R_E\|_{0;E} \right\} \\
\geq & (\gamma_1 c_4 - 2d\gamma_2 c_8^2 c_6^{-1}) \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \alpha_K^2 \|R_K\|_{0;K}^2 \\
& + \frac{1}{2}\gamma_2 c_6 \sum_{E\in\widetilde{\mathcal{E}}_{h,n}} \lambda^{-1/2}\alpha_E \|R_E\|_{0;E}^2 \\
\geq & \min\left\{ \gamma_1 c_4 - 2d\gamma_2 c_8^2 c_6^{-1}, \frac{1}{2}\gamma_2 c_6 \right\} \left(\eta_h^n\right)^2 .
\end{aligned}
\tag{5.16}
$$

Now we choose

$$
\gamma_2 = \frac{2}{c_6} \quad \text{and} \quad \gamma_1 = \frac{1}{c_4}\left(1 + \frac{4dc_8^2}{c_6^2}\right).
$$

This choice gives

$$
\min\left\{ \gamma_1 c_4 - 2d\gamma_2 c_8^2 c_6^{-1}, \frac{1}{2}\gamma_2 c_6 \right\} = 1.
$$

Estimates (5.16) and (5.15) and equation (5.11) now imply the lower bound (5.10).
□

6. Estimation of the temporal residual

The following lemma provides us with sharp upper and lower bounds for the temporal residual.

6.1 Lemma. *For every integer n between* 1 *and* N *the temporal residual is bounded from above by*

$$\begin{aligned}&\left\{\int_{t_{n-1}}^{t_n} |||R_\tau(u_{h,\tau})(.,s)|||_*^2 ds\right\}^{1/2}\\&\qquad\le\sqrt{\frac{1}{3}(1+\kappa^2)}\tau_n^{1/2}\left\{|||u_h^n-u_h^{n-1}|||^2+|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*^2\right\}^{1/2}.\end{aligned}\tag{6.1}$$

For every interval $(t_{n-1},t_n]$ *and every real number* δ *larger than* 0 *and less than* 1 *there is a function* $z_{n,\delta}\in L^2(0,T;H_D^1(\Omega))$ *such that the temporal residual is bounded from below by*

$$\begin{aligned}&\int_{t_{n-1}}^{t_n}\langle R_\tau(u_{h,\tau})(.,s),z_{n,\delta}(.,s)\rangle ds\\&\qquad\ge\frac{\delta}{12(\delta+\kappa^2)}\tau_n\left\{|||u_h^n-u_h^{n-1}|||^2+\delta|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*^2\right\},\\&\left\{\int_{t_{n-1}}^{t_n}|||z_{n,\delta}(.,s)|||^2ds\right\}^{1/2}\\&\qquad\le\sqrt{\frac{2}{3}}\tau_n^{1/2}\left\{|||u_h^n-u_h^{n-1}|||^2+|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*^2\right\}^{1/2}.\end{aligned}\tag{6.2}$$

Here, κ *is the constant of Assumption (A2) of §1.*

Proof. Since the function $u_{h,\tau}$ is piecewise affine with respect to time, we have on each time interval $[t_{m-1},t_m]$

$$\theta u_h^m+(1-\theta)u_h^{m-1}-u_{h,\tau}=\left[\theta-\frac{t-t_{m-1}}{\tau_m}\right](u_h^m-u_h^{m-1}).$$

For abbreviation we define for each m between 1 and N the quantity $r_m\in H^{-1}(\Omega)$ by

$$\begin{aligned}\langle r_m,v\rangle=&(D\nabla(u_h^m-u_h^{m-1}),\nabla v)+(\underline{c}\cdot\nabla(u_h^m-u_h^{m-1}),v)\\&+(r(u_h^m-u_h^{m-1}),v)\qquad\forall v\in H_D^1(\Omega).\end{aligned}$$

Then we obtain the following representation of the temporal residual

$$R_\tau(u_{h,\tau})=\left[\theta-\frac{t-t_{m-1}}{\tau_m}\right]r_m\qquad\text{on }(t_{m-1},t_m],\ 1\le m\le N.\tag{6.3}$$

A straightforward calculation gives

$$\int_{t_{m-1}}^{t_m}\left[\theta-\frac{t-t_{m-1}}{\tau_m}\right]^2dt=\tau_m\frac{1}{3}\left[\theta^3+(1-\theta)^3\right]$$

and consequently

$$\frac{1}{12}\tau_m \le \int_{t_{m-1}}^{t_m} \left[\theta - \frac{t-t_{m-1}}{\tau_m}\right]^2 dt \le \frac{1}{3}\tau_m. \tag{6.4}$$

From inequality (2.8) we conclude that

$$\begin{aligned} |||r_m|||_* \le& \kappa|||u_h^m - u_h^{m-1}||| + |||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_* \\ \le& \sqrt{1+\kappa^2}\left\{|||u_h^m - u_h^{m-1}|||^2 + |||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_*^2\right\}^{1/2}. \end{aligned} \tag{6.5}$$

Equation (6.3) and inequalities (6.4) and (6.5) prove the upper bound (6.1).
Due to the definition (2.3) of $|||.|||_*$ there is for each $\delta \in (0,1)$ a function $\varphi_{m,\delta} \in H_D^1(\Omega)$ with

$$\begin{aligned} |||\varphi_{m,\delta}||| &= |||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_*, \\ (\underline{c}\cdot\nabla(u_h^m - u_h^{m-1}), \varphi_{m,\delta}) &\ge \delta|||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_*^2. \end{aligned}$$

We set

$$\zeta_{m,\delta} = (u_h^m - u_h^{m-1}) + \gamma\varphi_{m,\delta}, \tag{6.6}$$

where γ is a constant that will be fixed below. Obviously we have

$$|||\zeta_{m,\delta}||| \le \max\{1,\gamma\}\left\{|||u_h^m - u_h^{m-1}||| + |||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_*\right\}.$$

Inequalities (2.7) and (2.8) on the other hand yield

$$\begin{aligned} \langle r_m, \zeta_{m,\delta}\rangle \ge& |||u_h^m - u_h^{m-1}|||^2 + \gamma\delta|||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_*^2 \\ &- \gamma\kappa|||u_h^m - u_h^{m-1}|||\ |||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_* \\ \ge& \left\{1 - \frac{1}{2}\gamma\delta^{-1}\kappa^2\right\}|||u_h^m - u_h^{m-1}|||^2 \\ &+ \frac{1}{2}\gamma\delta|||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_*^2. \end{aligned}$$

Now we choose

$$\gamma = \frac{2\delta}{\delta+\kappa^2}$$

and obtain

$$\begin{aligned} |||\zeta_{m,\delta}||| \le& |||u_h^m - u_h^{m-1}||| + |||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_*, \\ \langle r_m, \zeta_{m,\delta}\rangle \ge& \frac{\delta}{\delta+\kappa^2}\Big\{|||u_h^m - u_h^{m-1}|||^2 + \delta|||\underline{c}\cdot\nabla(u_h^m - u_h^{m-1})|||_*^2\Big\}. \end{aligned} \tag{6.7}$$

Equation (6.3) and estimates (6.4), (6.7) show that the function

$$z_{m,\delta} = \left[\theta - \frac{t-t_{m-1}}{\tau_m}\right]\zeta_{m,\delta}$$

yields the lower bounds (6.2). □

6.2 Remark. The arguments used to prove estimates (6.7) yield the inf-sup condition

$$\inf_{v\in H^1_D(\Omega)} \sup_{w\in H^1_D(\Omega)} \frac{(D\nabla v,\nabla w)+(\underline{c}\cdot\nabla v,w)+(rv,w)}{\{|||v|||^2+|||\underline{c}\cdot\nabla v|||_*^2\}^{1/2}|||w|||} \geq \frac{1}{\sqrt{2}}\frac{1}{1+\kappa^2}.$$

A similar result is established in [7]. There, however, the infimum and supremum are both taken with respect to the same norm which is defined by interpolation between $|||.|||$ and $|||.|||+|||\underline{c}\cdot\nabla.|||_*$. But the present result is better suited for our purposes.

7. A preliminary a posteriori error estimate

The following lemma provides us with a posteriori error bounds which are robust in the sense described in the Introduction. However, they are not suited for practical computations since they involve terms of the form $|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*$. In the next section we will bound these terms by computable quantities. Recall that $H^{-1}(\Omega)$ is equipped with $|||.|||_*$.

7.1 Lemma. *The error between the solution u of problem (2.6) and the solution $u_{h,\tau}$ of problems (3.1), (3.2) is bounded from above by*

$$\begin{aligned}\|u-&u_{h,\tau}\|_{X(0,T)}\leq\\ &c^*\Bigg\{\|u_0-\pi_0u_0\|_0^2\\ &\qquad+\sum_{n=1}^N\tau_n\Big[(\eta_h^n)^2+|||u_h^n-u_h^{n-1}|||^2+|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*^2\Big]\\ &\qquad+\big\|f-f_{h,\tau}\big\|^2_{L^2(0,T;H^{-1}(\Omega))}+\big\|g-g_{h,\tau}\big\|^2_{L^2(0,T;H^{-1/2}(\Gamma_N))}\Bigg\}^{1/2}\end{aligned}\tag{7.1}$$

and on each interval $(t_{n-1},t_n]$, $1\leq n\leq N$, from below by

$$\begin{aligned}&\tau_n^{1/2}\Big\{(\eta_h^n)^2+|||u_h^n-u_h^{n-1}|||^2+|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*^2\Big\}^{1/2}\\ \leq&c_*\Big\{\|u-u_{h,\tau}\|^2_{X(t_{n-1},t_n)}\\ &\quad+\big\|f-f_{h,\tau}\big\|^2_{L^2(t_{n-1},t_n;H^{-1}(\Omega))}+\big\|g-g_{h,\tau}\big\|^2_{L^2(t_{n-1},t_n;H^{-1/2}(\Gamma_N))}\Big\}^{1/2}.\end{aligned}\tag{7.2}$$

The quantity η_h^n is defined in equation (5.8). The constants c^* and c_* depend on the ratios h_K/ρ_K. The constant c^* in addition depends on the ratios $h_{K'}/h_K$. The constant c_* in addition depends on the maximum of the polynomial degrees of the finite *element functions and on the constant κ of Assumption (A2) of §1. All constants are independent of the final time T, and the parameters λ, $|\underline{c}|$, and r.*

Proof. The upper bound (7.1) follows from estimates (4.3), (5.9), and (6.1) and the decomposition (4.11) of the residual.

For the proof of the lower bound (7.2) we choose an integer n between 1 and N and a real number δ larger than 0 and less than 1. First we insert the function $z_{n,\delta}$ of Lemma 6.1 into the representation (4.11) of the residual. Estimates (6.2), (5.9), and (4.2) then imply

$$\begin{aligned} &\frac{\delta}{12(\delta+\kappa^2)}\tau_n\Big\{|||u_h^n-u_h^{n-1}|||^2+\delta|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*^2\Big\} \\ &\le\int_{t_{n-1}}^{t_n}\langle R_\tau(u_{h,\tau})(.,s),z_{n,\delta}(.,s)\rangle ds \\ &=\int_{t_{n-1}}^{t_n}\langle R(u_{h,\tau})(.,s)-R_D(u_{h,\tau})(.,s)-R_h(u_{h,\tau})(.,s),z_{n,\delta}(.,s)\rangle ds \end{aligned}$$

and

$$\begin{aligned} &\int_{t_{n-1}}^{t_n}\langle R(u_{h,\tau})(.,s)-R_D(u_{h,\tau})(.,s)-R_h(u_{h,\tau})(.,s),z_{n,\delta}(.,s)\rangle ds \\ &\le\sqrt{\frac{2}{3}}\tau_n^{1/2}\left\{|||u_h^n-u_h^{n-1}|||^2+|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*^2\right\}^{1/2} \\ &\quad\cdot\Big\{\sqrt{1+\kappa^2}\|u-u_{h,\tau}\|_{X(t_{n-1},t_n)}+\big\|f-f_{h,\tau}\big\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \\ &\qquad+\big\|g-g_{h,\tau}\big\|_{L^2(t_{n-1},t_n;H^{-1/2}(\Gamma_N))}+c^\dagger\tau_n^{1/2}\eta_h^n\Big\}. \end{aligned}$$

Since $\delta\in(0,1)$ was arbitrary and since $\sqrt{\frac{2}{3}}\le 1$ this yields the estimate

$$\begin{aligned} &\tau_n^{1/2}\Big\{|||u_h^n-u_h^{n-1}|||^2+|||\underline{c}\cdot\nabla(u_h^n-u_h^{n-1})|||_*^2\Big\}^{1/2} \\ &\quad\le c'\Big\{\sqrt{1+\kappa^2}\|u-u_{h,\tau}\|_{X(t_{n-1},t_n)}+\big\|f-f_{h,\tau}\big\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \\ &\qquad+\big\|g-g_{h,\tau}\big\|_{L^2(t_{n-1},t_n;H^{-1/2}(\Gamma_N))}+c^\dagger\tau_n^{1/2}\eta_h^n\Big\} \end{aligned} \tag{7.3}$$

with $c'=12(1+\kappa^2)$.

Next we insert the function $(\alpha+1)\left(\frac{t-t_{n-1}}{\tau_n}\right)^\alpha w_n$ into the representation (4.11) of the residual. Here w_n is the function of Lemma 5.4 and α denotes a non-negative constant that will be determined below. Estimate (5.10) and the decomposition (4.11) of the residual then yield

$$\begin{aligned} \tau_n\left(\eta_h^n\right)^2&\le\int_{t_{n-1}}^{t_n}(\alpha+1)\left(\frac{t-t_{n-1}}{\tau_n}\right)^\alpha\langle R_h(u_{h,\tau}),w_n\rangle dt \\ &=\int_{t_{n-1}}^{t_n}(\alpha+1)\left(\frac{t-t_{n-1}}{\tau_n}\right)^\alpha\langle R(u_{h,\tau})-R_D(u_{h,\tau})-R_\tau(u_{h,\tau}),w_n\rangle dt. \end{aligned}$$

Since

$$\int_{t_{n-1}}^{t_n} (\alpha+1)^2 \left(\frac{t-t_{n-1}}{\tau_n}\right)^{2\alpha} dt = \frac{(\alpha+1)^2}{2\alpha+1}\tau_n \le (2\alpha+1)\tau_n,$$

estimates (4.2) and (5.10) imply that

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} &(\alpha+1)\left(\frac{t-t_{n-1}}{\tau_n}\right)^{\alpha} \langle R(u_{h,\tau}) - R_D(u_{h,\tau}), w_n\rangle dt \\
&\le \sqrt{2\alpha+1} c_\dagger \tau_n^{1/2} \eta_h^n \Big\{ \sqrt{1+\kappa^2} \|u-u_{h,\tau}\|_{X(t_{n-1},t_n)} \\
&\qquad + \left\|f - f_{h,\tau}\right\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \\
&\qquad + \left\|g - g_{h,\tau}\right\|^2_{L^2(t_{n-1},t_n;H^{-1/2}(\Gamma_N))} \Big\}.
\end{aligned}$$

Since

$$\int_{t_{n-1}}^{t_n} (\alpha+1)\left(\frac{t-t_{n-1}}{\tau_n}\right)^{\alpha} \left[\theta - \frac{t-t_{n-1}}{\tau_n}\right] dt = \left(\theta - \frac{\alpha+1}{\alpha+2}\right)\tau_n$$

and $\sqrt{\frac{1}{3}(1+\kappa^2)} \le \kappa$, we conclude from estimates (5.10) and (6.1) that

$$\begin{aligned}
\int_{t_{n-1}}^{t_n} &(\alpha+1)\left(\frac{t-t_{n-1}}{\tau_n}\right)^{\alpha} \langle R_\tau(u_{h,\tau}), w_n\rangle dt \\
&\le \left|\theta - \frac{\alpha+1}{\alpha+2}\right| \kappa c_\dagger \eta_h^n \tau_n \Big\{ |||u_h^n - u_h^{n-1}||| + |||\underline{c}\cdot\nabla(u_h^n - u_h^{n-1})|||_* \Big\}.
\end{aligned}$$

Combining these estimates and inserting inequality (7.3) we arrive at the estimate

$$\begin{aligned}
\tau_n \left(\eta_h^n\right)^2 \le &\left|\theta - \frac{\alpha+1}{\alpha+2}\right| c_\dagger c'' \tau_n \left(\eta_h^n\right)^2 \\
&+ \tau_n^{1/2} \eta_h^n c_\dagger c''' \left[\sqrt{2\alpha+1} + \left|\theta - \frac{\alpha+1}{\alpha+2}\right|\right] \\
&\cdot \Big\{ \|u - u_{h,\tau}\|_{X(t_{n-1},t_n)} \\
&\quad + \left\|f - f_{h,\tau}\right\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \\
&\quad + \left\|g - g_{h,\tau}\right\|^2_{L^2(t_{n-1},t_n;H^{-1/2}(\Gamma_N))} \Big\}
\end{aligned} \tag{7.4}$$

with constants c'' and c''' that only depend on the constants κ, $c_\dagger$, and $c^\dagger$.
Now we choose the parameter α such that the first term on the right-hand side of inequality (7.4) is balanced by the term on the left-hand side. In case of the Crank-Nicolson scheme, i.e. $\theta = \frac{1}{2}$, this is obvious: We have to choose $\alpha = 0$. In the remaining cases $\frac{1}{2} < \theta \le 1$ we set

$$\alpha = \frac{2c_\dagger c''(2\theta-1)}{2c_\dagger c''(1-\theta)+1}.$$

Since we may assume that $c_\dagger c'' \geq 1$ this implies

$$\frac{\alpha+1}{\alpha+2} \leq \theta \quad \text{and} \quad \left|\theta - \frac{\alpha+1}{\alpha+2}\right| c_\dagger c'' \leq \frac{1}{2}.$$

Estimate (7.4) therefore takes the form

$$\begin{aligned} \tau_n^{1/2}\eta_h^n \leq & c\Big\{ \|u-u_{h,\tau}\|_{X(t_{n-1},t_n)} \\ & + \big\|f-f_{h,\tau}\big\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \\ & + \big\|g-g_{h,\tau}\big\|^2_{L^2(t_{n-1},t_n;H^{-1/2}(\Gamma_N))}\Big\} \end{aligned} \tag{7.5}$$

with a constant c that only depends on the constants κ, $c_\dagger$, and $c^\dagger$. Estimates (7.3) and (7.5) obviously imply the lower bound (7.2). □

8. A robust a posteriori error estimator

In this section we derive computable and robust bounds for the terms $|||\underline{c}\cdot\nabla(u_h^n - u_h^{n-1})|||_*$ in Lemma 7.1. To this end we must distinguish two cases:
small convection: $|\underline{c}| \leq c_c\lambda^{1/2}\max\{\lambda, r\}^{1/2}$ with a constant c_c of moderate size;
large convection: $|\underline{c}| \gg \lambda^{1/2}\max\{\lambda, r\}^{1/2}$.

In the first case the situation is quite simple since we can bound $|||\underline{c}\cdot\nabla(u_h^n - u_h^{n-1})|||_*$ by $|||u_h^n - u_h^{n-1}|||$ times a constant of moderate size. To be precise denote by c_Ω the constant in the Poincaré inequality

$$\|w\|_0 \leq c_\Omega\|\nabla w\|_0 \quad \forall w \in H^1_D(\Omega).$$

This estimate and the definition (2.2) of the energy norm yield for all $v, w \in H^1_D(\Omega)$ the estimate

$$\begin{aligned} (\underline{c}\cdot\nabla v, w) \leq & |\underline{c}|\|\nabla v\|_0\|w\|_0 \\ \leq & |\underline{c}|\lambda^{-1/2}|||v|||\min\{r^{-1/2}, c_\Omega\lambda^{-1/2}\}|||w||| \\ \leq & \max\{1, c_\Omega\}|\underline{c}|\lambda^{-1/2}\min\{r^{-1/2}, \lambda^{-1/2}\}|||v|||\ |||w||| \\ = & \max\{1, c_\Omega\}|\underline{c}|\lambda^{-1/2}\max\{r, \lambda\}^{-1/2}|||v|||\ |||w||| \\ \leq & \max\{1, c_\Omega\}c_c|||v|||\ |||w|||. \end{aligned}$$

Recalling the definition (2.3) of the dual norm this implies that

$$|||\underline{c}\cdot\nabla(u_h^n - u_h^{n-1})|||_* \leq \max\{1, c_\Omega\}c_c|||u_h^n - u_h^{n-1}|||. \tag{8.1}$$

When bounding the convection term in (7.1) using estimate (8.1) and dropping the convection term in (7.2), we arrive at the following result for the case of small convection:

8.1 Theorem. *Assume that* $|\underline{c}| \le c_c \lambda^{1/2} \max\{\lambda, r\}^{1/2}$ *with a constant* c_c *of moderate size. Then the error between the solution* u *of problem (2.6) and the solution* $u_{h,\tau}$ *of problems (3.1), (3.2) is bounded from above by*

$$\begin{aligned}
&\|u-u_{h,\tau}\|_{X(0,T)} \le \\
&\quad \hat{c}^* \Bigg\{ \|u_0 - \pi_0 u_0\|_0^2 \\
&\qquad + \sum_{n=1}^{N} \tau_n \Big[(\eta_h^n)^2 + |||u_h^n - u_h^{n-1}|||^2 \Big] \\
&\qquad + \big\|f - f_{h,\tau}\big\|^2_{L^2(0,T;H^{-1}(\Omega))} + \big\|g - g_{h,\tau}\big\|^2_{L^2(0,T;H^{-1/2}(\Gamma_N))} \Bigg\}^{1/2}
\end{aligned} \tag{8.2}$$

and on each interval $(t_{n-1}, t_n]$, $1 \le n \le N$, *from below by*

$$\begin{aligned}
\tau_n^{1/2} &\Big\{ (\eta_h^n)^2 + |||u_h^n - u_h^{n-1}|||^2 \Big\}^{1/2} \\
&\le c_* \Big\{ \|u - u_{h,\tau}\|^2_{X(t_{n-1},t_n)} \\
&\qquad + \big\|f - f_{h,\tau}\big\|^2_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \\
&\qquad + \big\|g - g_{h,\tau}\big\|^2_{L^2(t_{n-1},t_n;H^{-1/2}(\Gamma_N))} \Big\}^{1/2}.
\end{aligned} \tag{8.3}$$

The quantity η_h^n *is defined in equation (5.8). The constants* $\hat{c}^*$ *and* c_* *depend on the ratios* h_K/ρ_K. *The constant* $\hat{c}^*$ *in addition depends on the ratios* $h_{K'}/h_K$ *and on the constants* c_c *and* c_Ω. *The constant* c_* *in addition depends on the maximum of the polynomial degrees of the finite element functions and on the constant* κ *of Assumption (A2) of §1. All constants are independent of the final time* T, *and the parameters* λ, $|\underline{c}|$, *and* r.

In the case of large convection, estimate (8.1) incorporates too large a constant. In this case we must bound the dual norms of the convection terms in a more sophisticated way. The idea is as follows: Due to the definition of the dual norm, these quantities equal the energy norm of the weak solutions of suitable stationary reaction-diffusion equations. These solutions are approximated by suitable finite element functions. The error of the approximations is estimated by robust error estimators for reaction-diffusion equations.

8.2 Lemma. *For every integer* n *between* 1 *and* N *set*

$$\widetilde{X}_{h,n} = \{v \in C(\Omega) : v_{|K} \in R_1(K) \text{ for all } K \in \widetilde{\mathcal{T}}_{h,n},\ v = 0 \text{ on } \Gamma_D\}$$

and denote by $\widetilde{u}_h^n \in \widetilde{X}_{h,n}$ *the unique solution of the discrete reaction-diffusion problem*

$$\lambda(\nabla \widetilde{u}_h^n, \nabla v_h) + r(\widetilde{u}_h^n, v_h) = (\underline{c} \cdot \nabla(u_h^n - u_h^{n-1}), v_h) \quad \forall v_h \in \widetilde{X}_{h,n}. \tag{8.4}$$

Define the error indicator $\widetilde{\eta}_h^n$ *by*

$$\begin{aligned} \widetilde{\eta}_h^n = \Big\{ & \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \alpha_K^2 \| \underline{c} \cdot \nabla(u_h^n - u_h^{n-1}) + \lambda \Delta \widetilde{u}_h^n - r \widetilde{u}_h^n \|_{0;K}^2 \\ & + \sum_{E \in \widetilde{\mathcal{E}}_{h,n} \setminus \Gamma_D} \lambda^{-1/2} \alpha_E \| [\underline{n}_E \cdot \nabla \widetilde{u}_h^n]_E \|_{0;E}^2 \Big\}^{1/2}. \end{aligned} \tag{8.5}$$

Then there are two constants $\widetilde{c}_\dagger$ *and* $\widetilde{c}^\dagger$ *which only depend on the ratios* h_K/ρ_K *such that the following estimates are valid*

$$\widetilde{c}_\dagger \left\{ |||\widetilde{u}_h^n||| + \widetilde{\eta}_h^n \right\} \le |||\underline{c} \cdot \nabla(u_h^n - u_h^{n-1})|||_* \le \widetilde{c}^\dagger \left\{ |||\widetilde{u}_h^n||| + \widetilde{\eta}_h^n \right\}. \tag{8.6}$$

Proof. We choose an integer n between 1 and N and keep it fixed in what follows. Denote by $\widetilde{U}^n \in H_D^1(\Omega)$ the unique solution of the stationary reaction-diffusion equation

$$\lambda(\nabla \widetilde{U}^n, \nabla v) + r(\widetilde{U}^n, v) = (\underline{c} \cdot \nabla(u_h^n - u_h^{n-1}), v) \quad \forall v \in H_D^1(\Omega).$$

The definitions (2.2) and (2.3) of the energy norm $|||.|||$ and of the dual norm $|||.|||_*$ respectively imply that

$$|||\widetilde{U}^n||| = |||\underline{c} \cdot \nabla(u_h^n - u_h^{n-1})|||_*.$$

Inserting $v_h = \widetilde{u}_h^n$ as a test function in the discrete problem (8.4) we obtain

$$|||\widetilde{u}_h^n||| \le |||\underline{c} \cdot \nabla(u_h^n - u_h^{n-1})|||_*.$$

The triangle inequality therefore yields

$$\frac{1}{3} \left\{ |||\widetilde{u}_h^n||| + |||\widetilde{U}^n - \widetilde{u}_h^n||| \right\} \le |||\underline{c} \cdot \nabla(u_h^n - u_h^{n-1})|||_* \le \left\{ |||\widetilde{u}_h^n||| + |||\widetilde{U}^n - \widetilde{u}_h^n||| \right\}.$$

Since $\underline{c} \cdot \nabla(u_h^n - u_h^{n-1})$ is a piecewise polynomial, we know from [9] that $\widetilde{\eta}_h^n$ yields upper and lower bounds for $|||\widetilde{U}^n - \widetilde{u}_h^n|||$ with multiplicative constants that only depend on the ratios h_K/ρ_K. This proves estimate (8.6). □

Combining Lemmas 7.1 and 8.2 we obtain the following result:

8.3 Theorem. *Assume that* $|\underline{c}| \gg \lambda^{1/2} \max\{\lambda, r\}^{1/2}$. *Then the error between the solution* u *of problem (2.6) and the solution* $u_{h,\tau}$ *of problems (3.1), (3.2) is bounded from above by*

$$\begin{aligned} \|u - u_{h,\tau}\|_{X(0,T)} \leq \\ \widetilde{c}^* \Big\{ & \|u_0 - \pi_0 u_0\|_0^2 \\ & + \sum_{n=1}^{N} \tau_n \Big[(\eta_h^n)^2 + |||u_h^n - u_h^{n-1}|||^2 + (\widetilde{\eta}_h^n)^2 + |||\widetilde{u}_h^n|||^2 \Big] \\ & + \|f - f_{h,\tau}\|^2_{L^2(0,T;H^{-1}(\Omega))} \\ & + \|g - g_{h,\tau}\|^2_{L^2(0,T;H^{-1/2}(\Gamma_N))} \Big\}^{1/2} \end{aligned} \tag{8.7}$$

and on each interval $(t_{n-1}, t_n]$, $1 \leq n \leq N$, *from below by*

$$\begin{aligned} \tau_n^{1/2} \Big\{ (\eta_h^n)^2 + & |||u_h^n - u_h^{n-1}|||^2 + (\widetilde{\eta}_h^n)^2 + |||\widetilde{u}_h^n|||^2 \Big\}^{1/2} \\ \leq \widetilde{c}_* \Big\{ & \|u - u_{h,\tau}\|^2_{X(t_{n-1},t_n)} \\ & + \|f - f_{h,\tau}\|^2_{L^2(t_{n-1},t_n;H^{-1}(\Omega))} \\ & + \|g - g_{h,\tau}\|^2_{L^2(t_{n-1},t_n;H^{-1/2}(\Gamma_N))} \Big\}^{1/2}. \end{aligned} \tag{8.8}$$

The quantity η_h^n *is defined in equation (5.8). The constants* $\widetilde{c}^*$ *and* $\widetilde{c}_*$ *depend on the ratios* h_K/ρ_K. *The constant* $\widetilde{c}^*$ *in addition depends on the ratios* $h_{K'}/h_K$. *The constant* $\widetilde{c}_*$ *in addition depends on the maximum of the polynomial degrees of the finite element functions and on the constant* κ *of Assumption (A2) of §1. All constants are independent of the final time* T, *and the parameters* λ, $|\underline{c}|$, *and* r.

8.4 Remark. Theorem 8.3 shows that the quantity $\tau_n^{1/2}\Big\{(\eta_h^n)^2 + |||u_h^n - u_h^{n-1}|||^2 + (\widetilde{\eta}_h^n)^2 + |||\widetilde{u}_h^n|||^2\Big\}^{1/2}$ is a robust error indicator in the sense described in the Introduction. The remaining terms on the right-hand side of estimates (8.7) and (8.8) are data errors. The term $\tau_n^{1/2}\eta_h^n$ can be interpreted as a spatial error indicator and can be used to monitor the space-adaptivity. The terms $\tau_n^{1/2}\Big\{|||u_h^n - u_h^{n-1}|||^2 + (\widetilde{\eta}_h^n)^2 + |||\widetilde{u}_h^n|||^2\Big\}^{1/2}$ on the other hand can be viewed as temporal error indicators and can be used to monitor the time-adaptivity. Theorem 8.1 shows that in the case of small convection the terms involving $\widetilde{\eta}_h^n$ and $\widetilde{u}_h^n$ can be dropped without loosing robustness. Thus the evaluation of the error estimator is much simpler in this case.

9. Variable coefficients

In this section we present the modifications that are necessary to adapt the previous analysis to equations with variable coefficients.

Assumptions (A1) – (A3) of §1 must be replaced by the following conditons on the coefficients:

(A1') The coefficients satisfy $D \in C(0,T;L^\infty(\Omega)^{d\times d})$, $\underline{c} \in C(0,T;W^{1,\infty}(\Omega)^d)$, $r \in C(0,T;L^\infty(\Omega))$.

(A2') The diffusion-coefficient D is symmetric, uniformly positive definite and uniformly isotropic, i.e.

$$\lambda = \inf_{0<t\le T, x\in\Omega} \min_{z\in\mathbb{R}^d\setminus\{0\}} \frac{z^T D(x,t) z}{z^T z} > 0$$

and

$$\kappa = \lambda^{-1} \sup_{0<t\le T, x\in\Omega} \max_{z\in\mathbb{R}^d\setminus\{0\}} \frac{z^T D(x,t) z}{z^T z} = O(1).$$

(A3') There is a constant $\beta \ge 0$ such that $r - \frac{1}{2}\operatorname{div}\underline{c} \ge \beta$ for almost all $x \in \Omega$ and $0 < t \le T$. Moreover there is a constant $c_r \ge 0$ of moderate size such that $\|r\|_{L^\infty(0,T;L^\infty(\Omega))} \le c_r\beta$.

With these assumptions the various regimes mentioned in §1 can now be characterized by:

dominant diffusion: $|\underline{c}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \le c_c\lambda$ and $\beta \le c_r'\lambda$ with constants of order 1;
dominant reaction: $|\underline{c}|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \le c_c\lambda$ and $\beta \gg \lambda$ with constant c_c of order 1;
dominant convection: $\beta \gg \lambda$.

In the definition (2.2) of the energy norm the quantity r must be replaced by the constant β. This definition of the energy norm is then used in the definition (2.3) of the dual norm and in the definitons of the norms of $H^{1/2}(\Gamma_N)$ and $H^{-1/2}(\Gamma_N)$. With these modifications estimates (2.7) and (2.8) remain valid.

In the finite element discretization (3.2) the coefficients must now be discretized in time by replacing D, $\underline{c}$, and r by $D^{n\theta} = \theta D(.,t_n) + (1-\theta)D(.,t_{n-1})$, $\underline{c}^{n\theta} = \theta\underline{c}(.,t_n) + (1-\theta)\underline{c}(.,t_{n-1})$, and $r^{n\theta} = \theta r(.,t_n) + (1-\theta)r(.,t_{n-1})$ respectively.

Lemma 4.1 remains unchanged. But in the definitions (4.8) and (4.9) of the temporal and spatial residuals the coefficients D, $\underline{c}$, and r are replaced by their time-discretizations $D^{n\theta}$, $\underline{c}^{n\theta}$, and $r^{n\theta}$ respectively. This gives rise to additional terms

$$((D^{n\theta} - D)\nabla u_{h,\tau}, \nabla v) + ((\underline{c}^{n\theta} - \underline{c})\cdot\nabla u_{h,\tau}, v) + ((r^{n\theta} - r)u_{h,\tau}, v)$$

in the definition (4.10) of the data residual. These terms introduce additional data errors

$$\lambda^{-1}\|(D^{n\theta} - D)\nabla u_{h.\tau}\|_{L^2(a,b;H^1_D(\Omega))} + \|(\underline{c}^{n\theta} - \underline{c})\cdot\nabla u_{h.\tau} + (r^{n\theta} - r)u_{h,\tau}\|_{L^2(a,b;H^{-1}(\Omega))}$$

with appropriate values of a and b on the right-hand sides of estimates (7.1), (7.2), (8.2), (8.3), (8.7), and (8.8).

In the definition (5.4) of the weights α_S the quantity r must be replaced by the constant β. In the definition (5.8) of the spatial error indicator η_h^n the coefficients D, $\underline{c}$, and r must be replaced by finite element approximations $D_h^{n\theta}$, $\underline{c}_h^{n\theta}$, and $r_h^{n\theta}$ of the time-discretizations $D^{n\theta}$, $\underline{c}^{n\theta}$, and $r^{n\theta}$. These finite element approximations are arbitrary; the simplest choice is the correponding L^2-projection onto the space of piecewise constant finite element functions. The spatial discretization of the coefficients gives rise to additonal elementwise data errors

$$
\begin{aligned}
D_K = \big\{ &- \operatorname{div}((D_h^{n\theta} - D^{n\theta})\nabla(\theta u_h^n + (1-\theta)u_h^{n-1})) \\
&+ (\underline{c}_h^{n\theta} - \underline{c}^{n\theta})\nabla \cdot (\theta u_h^n + (1-\theta)u_h^{n-1}) \\
&+ (r_h^{n\theta} - r^{n\theta})(\theta u_h^n + (1-\theta)u_h^{n-1})\big\}_{|K},
\end{aligned}
$$

for all $K \in \widetilde{\mathcal{T}}_{h,n}$, $1 \le n \le N$, and edge- respectively facewise data errors

$$
D_E = \big[\underline{n}_E \cdot ((D_h^{n\theta} - D^{n\theta})\nabla(\theta u_h^n + (1-\theta)u_h^{n-1}))\big]_E
$$

for all $E \in \widetilde{\mathcal{E}}_{h,n}$, $1 \le n \le N$. These data errors introduce an additional data error indicator

$$
\Theta_h^n = \left\{ \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \alpha_K^2 \|D_K\|_{L^2(K)}^2 + \sum_{E \in \widetilde{\mathcal{E}}_{h,n}} \varepsilon^{-1/2} \alpha_E \|D_E\|_{L^2(E)}^2 \right\}^{1/2}
$$

on the right-hand side of estimates (5.9) and (5.10). This data error indicator introduces an additional term

$$
\sum_{n=1}^{N} \tau_n \big(\Theta_h^n\big)^2
$$

on the right-hand sides of estimates (7.1), (8.2), and (8.7) and an additional term

$$
\tau_n \big(\Theta_h^n\big)^2
$$

on the right-hand sides of estimates (7.2), (8.3), and (8.8).

In equations (8.4) and (8.5) the quantities r and $\underline{c}$ must be replaced by β and $\underline{c}_h^{n\theta}$, respectively.

10. References

[1] R. A. Adams: Sobolev Spaces. Academic Press, New York, 1975

[2] H. Amann: Linear and Quasilinear Parabolic Problems. Volume I: Abstract Linear Theory. Birkhäuser, Basel, 1995

[3] C. Bernardi, R. Verfürth: Adaptive finite element methods for elliptic equations with non-smooth coefficients. Numer. Math. 85, 579 – 608 (2000)

[4] R. Dautray and J.-L. Lions: Mathematical Analysis and Numerical Methods for Science and Technology. Springer, Berlin - Heidelberg - New York, 1992

[5] L. P. Franca, S. L. Frey, T. J. R. Hughes: Stabilized finite element methods I: Application to the advective-diffusive model. Comput. Math. Appl. Mech. Engrg. 95, 253 – 271 (1992)

[6] T. J. R. Hughes, A. Brooks: Streamline upwind/ Petrov Galerkin formulations for the convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. Comp. Meth. Appl. Mech. Engrg. 54, 199 – 259 (1982)

[7] G. Sangalli: A uniform analysis of non-symmetric and coercive linear operators. Numer. Math. 97, 779 – 796 (2004)

[8] R. Verfürth: A posteriori error estimates for nonlinear problems: $L^r(0,T;W^{1,\rho}(\Omega))$–error estimates for finite element discretizations of parabolic equations. Numer. Meth. for PDE 14, 487 – 518 (1998)

[9] ———: Robust a posteriori error estimators for a singularly perturbed reaction-diffusion equation. Numer. Math. 78, 479 – 493 (1998)

[10] ———: Error estimates for some quasi-interpolation operators. Modél. Math. et Anal. Numér. 33, 695 – 713 (1999)

[11] ———: A posteriori error estimates for finite element discretizations of the heat equation. Calcolo 40, 195 – 212 (2003)

[12] ———: Robust a posteriori error estimates for non-stationary convection-diffusion equations. Report 333, Ruhr-Universität Bochum, 2004 (submitted to SIAM J. Numer. Anal.)

[13] ———: A posteriori error estimates for non-linear parabolic equations. (in preparation)